Proof of Theorem 19

Proof of (2)

Suppose that (2) does not hold, that is,

$$C^A - P^A - S(0) + K < 0$$

Then,

- At time $t=0$
  - Sell a put, buy a call and sell short a share, financing the transactions in the money market.
- If the American put is exercised at time $0 \leq t \leq T$
  - Borrow K from the money market and buy a share for K. This settles the short position on the American put.
  - Return the share to the owner, closing the short position in the share.
  - We still have the call option, which has nonnegative value.
  - We close the money market position.

The final balance of this strategy is the value of the call at time t and the amount

$$\underbrace{\left(P^A - C^A + S(0)\right)}_{K} e^{rt} - K > K e^{rt} - K > 0$$

* If the American put is not exercised.
  - We buy the share for K, exercising the call option at time T.
  - We close the short sale of the stock.
  - We close the money market position.

The final balance of this strategy

$$\left(P^A - C^A + S(0)\right) e^{rT} - K > K e^{rT} - K > 0$$

Proof of Proposition 20

Proof of (iii)

Suppose that $C^E \geq S(0)$. Then, we can sell the option and buy the stock, investing the balance on the money market. At time T, we sell the stock for $\min(S(T), K)$, settling the call option.

Our risk-less profit will be

$$\left(C^E - S(0)\right) e^{rT} + \min\left(S(T), K\right) > 0$$

Then, we have an arbitrage opportunity

On the other hand, from the put-call parity, and the bound $P^E \geq 0$, we obtain the lower bound

$$C^E = S(0) - Ke^{-rT} + \underbrace{P^E}_{\geq 0} \geq S(0) - Ke^{-rT}$$

Proof of (12)

From the put-call parity, and taking into account that $C^E < S(0)$, we get

$$P^E = \underbrace{C^E - S(0)}_{<0} + Ke^{-rT} < Ke^{-rT}$$

Moreover, from the put-call parity again, and using that $C^E \geq 0$, we get

$$P^E = \underbrace{C^E}_{\geq 0} - S(0) + Ke^{-rT} \geq Ke^{-rT} - S(0)$$

Proof of Theorem 21

We already know that $C^A \geq C^E$.

Suppose that $C^A > C^E$. Then, sell an American call and buy a European call, investing the balance in the money market.

- If the American call is exercised at $t \leq T$.
  - Borrow a share and sell it for K, closing the short position on the American call.
  - Invest K in the money market.
  - At time T, we use the European call to buy the share for K and close the short position in the stock.

The risk-less profit will be

$$\left(C^A - C^E\right) e^{rT} + K e^{r(T-t)} - K > 0$$

- If the American call is not exercised, You will end up with the European option (non-negative value) and a risk-less profit of $\left(C^A - C^E\right) e^{rT} > 0$.

Proof of Proposition 29

1. Obvious.

2. By the put-call parity we have

$$C^E(K_1) - P^E(K_1) = S(0) - K_1 e^{-rT}$$

$$C^E(K_2) - P^E(K_2) = S(0) - K_2 e^{-rT}$$

Subtracting the second equation from the first we get

$$C^E(K_1) - C^E(K_2) + P^E(K_2) - P^E(K_1) = (K_2 - K_1) e^{-rT}$$

Since both terms on the left hand side are positive, each one is strictly smaller than the right hand side.

3. Define $K = \alpha K_1 + (1-\alpha) K_2$. Suppose that

$$C^E(K) > \alpha C^E(K_1) + (1-\alpha) C^E(K_2).$$

We sell an option with strike K, buy α options with strike K_1 and $1-\alpha$ options with strike K_2 initial the balance

$$C^E(K) - \alpha C^E(K_1) - (1-\alpha) C^E(K_2) > 0$$

risk free.

If the option with strike K is exercised (at time T) we will have to pay $(S(T) - K)^+$. We can raise the amount

$$\alpha (S(T) - K_1)^+ + (1-\alpha)(S(T) - K_2)^+$$

by exercising the long position on the options.

The risk-less profit follows from the inequality

$$(S(T)-K)^+ \le \alpha (S(T)-K_1)^+ + (1-\alpha)(S(T)-K_2)^+,$$

which is left as exercise

Convexity for put options follows from the convexity of a call option, the put-call parity and the fact that the sum of convex function is convex.

Proof of Proposition 25

2. Suppose that $C^E(S_1) \geq C^E(S_2)$ for some $x_1 S(0) = S_1 < S_2 = x_2 S(0)$.

Sell a call on a portfolio with x_1 shares and buy a call on a portfolio with x_2 shares (same K, T). Invest the obtained amount $C^E(S_1) - C^E(S_2) > 0$ risk free.

Since $x_1 < x_2$, the payoffs satisfy

$$(x_1 S(T) - K)^+ \leq (x_2 S(T) - K)^+$$

with strict inequality if $K < x_2 S(T)$.

If the bought option is exercised at time T, then we use the other option to cover our liability and we are still left with a non-negative profit.

2. Follows by an analogous argument as the one used in Prop. 24.2., based on put-call parity.

3. The same argument as Prop. 24.3., but selling a call option on a portfolio of $X := \alpha x_1 + (1-\alpha) x_2$ and buying call options on a portfolio with x_1 shares and a portfolio with x_2 shares.

Proof of Proposition 25

1. Obvious.

2. We will only prove $K_1 < K_2$

$$C^A(K_1) - C^A(K_2) \leq (K_2 - K_1)$$

Suppose that one has that

$$C^A(K_1) - C^A(K_2) \geq K_2 - K_1 \qquad K_1 < K_2$$

then, we sell a call with strike K_2 and buy a call with strike K_2, investing $C^A(K_1) - C^A(K_2)$ with free. If the sold call is exercised at time $t \leq T$, we exercise the other call and we get

$$(S(t) - K_2)^+ - (S(t) - K_1)^+ \geq -(K_2 - K_1)$$

with strict inequality if $S(t) < K_2$.

If we combine this payoff with the

risk free investment, which amount to at least $(K_2 - K_1)e^{-rt}$, we end up with a non-negative amount of money.

3. Same strategy as European option, but taking into account possible early exercise.