Lecture 4

Example 1

Consider the LP problem

$$\max \; J(x) = 3x_1 + 2x_2$$

$$\text{subject to} \quad -x_1 + 3x_2 \leq 12$$

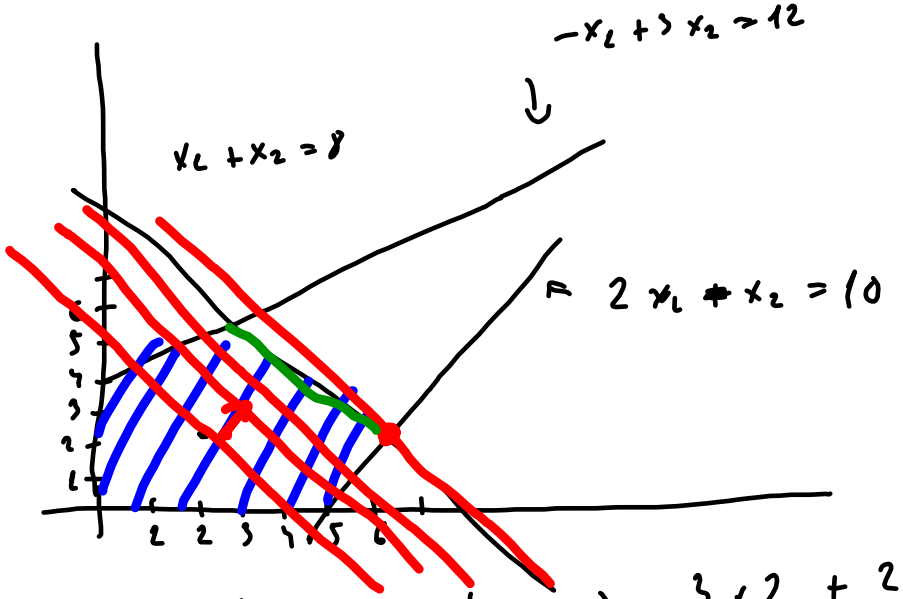
$$x_1 + x_2 \leq 8$$

$$2x_1 - x_2 \leq 10$$

$$x_1 \geq 0 \,, \quad x_2 \geq 0$$

Here, $c = (3,2)^T$, $b = (12, 8, 10)^T$, $A = \begin{pmatrix} -1 & 3 \\ 1 & 1 \\ 2 & -1 \end{pmatrix}$

Moreover, one can prove that the optimal solution is $\hat{x} = (6,2)^T$ and the optimal value is $\hat{J} := J(\hat{x}) = 22$



$(x_1, x_2) = (2, 2) \Rightarrow 3 \cdot 2 + 2 \cdot 2 = 10$ A lower bound for the optimal value.

$3x_1 + 2x_2 = 10 \Rightarrow x_2 = 5 - \frac{3}{2} x_1$

$x_1 = 0 \Rightarrow x_2 = 5$

To find an upper bound for the optimal value we can try conic linear combinations (i.e. linear combination with non-negative scalars) of the constraints. In this way the inequalities are not reversed

For example consider

$$\begin{aligned} 2\cdot(-x_1+3x_2) &\le 2\cdot 12 \\ 1\cdot(x_1+x_2) &\le 1\cdot 8 \\ 3\cdot(2x_1-x_2) &\le 3\cdot 10 \\ \hline 5x_1+4x_2 &\le 62 \end{aligned}$$

Since $$f(x) = 3x_1+2x_2 \le 5x_1+4x_2 \le 62$$

we obtain the upper bound $f(x)\le 62$.

We can use this procedure to get the best upper bound

Take $y_1, y_2, y_3 \ge 0$ and compute

$$\begin{aligned} y_1\cdot(-x_1+3x_2) &\le y_1\cdot 12 \\ y_2\cdot(x_1+x_2) &\le y_2\cdot 8 \\ y_3\cdot(2x_1-x_2) &\le y_3\cdot 10 \end{aligned}$$

which added yield

$$(-y_1 + y_2 + y_3)x_1 + (3y_1 + y_2 - y_3)x_2 \leq 12y_1 + 8y_2 + 10y_3$$

Since $z(x) = 3x_1 + 2x_2$, we take y_1, y_2, y_3 such that

$$-y_1 + y_2 + 2y_3 \geq 3 \quad \text{and} \quad 3y_1 + y_2 - y_3 \geq 2$$

$$z(x) = 3x_1 + 2x_2 \leq (-y_1 + y_2 + 2y_3)x_1 + (3y_1 + y_2 - y_3)x_2$$

$$\leq 12y_1 + 8y_2 + 10y_3$$

Finally to get the best upper bound we can solve the following LP.

$$\begin{aligned} \min \; & z(x) = 12y_1 + 8y_2 + 10y_3 \\ \text{subj. to} \; & -y_1 + y_2 + 2y_3 \geq 3 \\ & 3y_1 + y_2 - y_3 \geq 2 \\ & y_1 \geq 0 \quad y_2 \geq 0 \end{aligned}$$

This is the dual problem.

Proof of Corollary 10

Take $T = \{0\}$ in Theorem 9.

Then $\exists\ H_{\tilde{a},\tilde{\alpha}}$ such that

a) $\exists\ \varepsilon > 0$, $S \subseteq H^-_{\tilde{a},\tilde{\alpha}-\varepsilon} \Leftrightarrow \tilde{a}^T x \le \tilde{\alpha} - \varepsilon, \forall x \in S$ (1)

$T \subseteq H^+_{\tilde{a},\tilde{\alpha}+\varepsilon} \Leftrightarrow \tilde{a}^T 0 \ge \tilde{\alpha} + \varepsilon$

$\Leftrightarrow \tilde{\alpha} \le -\varepsilon$

$\Rightarrow \tilde{\alpha} < 0$ (2)

Take $a = -\tilde{a}$, then from (1) and (2) we get

$a^T x \ge -\tilde{\alpha} + \varepsilon \underset{\varepsilon>0}{>} -\tilde{\alpha} \underset{!!}{=} \alpha > 0, \forall x \in S.$

b) $\exists\ \varepsilon > 0$, $S \subseteq H^+_{\hat{a},\hat{\alpha}+\varepsilon} \Leftrightarrow \hat{a}^T x \ge \hat{\alpha} + \varepsilon, \forall x \in S$ (3)

$T \subseteq H^-_{\hat{a},\hat{\alpha}-\varepsilon} \Leftrightarrow \hat{a}^T 0 \le \hat{\alpha} - \varepsilon$

$\Leftrightarrow \hat{\alpha} \ge \varepsilon$

$\Rightarrow \hat{\alpha} > 0$ (4)

Take $a = \tilde{a}$, then from (3) and (4) we get

$a^T x \ge \tilde{\alpha} + \varepsilon \underset{\varepsilon>0}{>} \tilde{\alpha} \underset{(4)}{>} 0 \quad x \in S$

Proof of Corollary 11

Let M stand for the set

$$M = K - V = \{ m \in \mathbb{R}^n : m = k - v, k \in K, v \in V \}$$

One can check that M is a convex, closed set.

Since $K \cap V = \emptyset \rightarrow 0 \notin M$

From Corollary 9 we obtain the existence of $a \in \mathbb{R}^n$ and $\alpha > 0$ such that

$$a^T m \geq \alpha \quad \forall m \in M$$

which implies

$$(*) \quad a^T k - a^T v \geq \alpha \quad \forall k \in K \text{ and } v \in V$$

We fix $k \in K$ and take λv instead of v in $(*)$ where $\lambda \in \mathbb{R}$ (recall that $\lambda v \in V$ again)

Since λ is arbitrary and $(*)$ holds, i.e.,

$$a^T K - \lambda a^T v \geq \alpha$$

It is obvious that $a^T v = 0 \quad \forall v \in V$ and $\alpha \leq 0$

We conclude that

$$a^T K \geq \alpha > 0 \quad \forall K \in \mathcal{K}$$