Proof of Theorem 37

Let Q be a RNPM. Then, for every trading strategy H, one has that

$$V(0) = V^*(0) = E_Q[V^*(0)] = E_Q[V^*(1) - G^*]$$

($V^*(0)$ is constant; $V^*(1) = V^*(0) + G^*$)

$$= E_Q[V^*(1)] - E_Q[G^*]$$

$$= E_Q[V^*(1)] - E_Q\left[\sum_{n=1}^{N} H_n \Delta S_n^*\right]$$

$$= E_Q[V^*(1)] - \sum_{n=1}^{N} H_n \underbrace{E_Q[\Delta S_n^*]}_{=\,0 \text{ since } Q \text{ is a RNPM}}$$

$$= E_Q[V^*(1)] = E_Q\left[\frac{V(1)}{B(1)}\right]$$

Since $\mathcal{F}$ a RNPM, LOP holds and we can apply Proposition 3C we know that

$$p(X) = V(0) = E_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\right] = E_{\mathbb{Q}}\left[\frac{X}{B(1)}\right] = E_{\mathbb{Q}}[X^*]$$

($\mathbb{Q}$: replicating)

where V is the value process of a trading strategy generating X

This computation shows that $E_{\mathbb{Q}}\left[X/B(1)\right]$ DOES NOT depend on the $\mathbb{Q}$ chosen, as for any $\mathbb{Q}$ we obtain the same value $V(0)$ (which only depends on the contingent claim).

Proof of Lemma 44

$\Rightarrow$ We have already proved that if $\exists H$ with $V(1) = X$ then $V(0) = E_Q[X/\beta(1)]$ for any $Q \in \mathcal{M}$, and $V(0)$ does not depend on Q.

$(\Leftarrow)$ We will show that if the claim X is not attainable then $E_Q[X/\beta(1)]$ does not take the same value for all $Q \in \mathcal{M}$.

X is not attainable $\Leftrightarrow$ $S(1,N)H = X$ $\not\exists$ solution.

By Fredholm's alternative, then $\exists \pi = (\pi_1, \dots, \pi_K)^T \in \mathbb{R}^K$ such that $\pi^T S(1,N) = 0$, $\pi^T X > 0$

Pick any $\hat{Q} \in \mathcal{M} \neq \emptyset$ and we choose $\lambda \neq 0$ small enough to ensure that

$$Q_K := \hat{Q}_K + \lambda \pi_K B(\mathbb{1}) > 0 \quad \forall K = 1, \dots K$$

Since $\pi^T S(h \mathbb{1}) = 0 \Rightarrow B(\mathbb{1})\pi_1 + \dots + B(\mathbb{1})\pi_K = 0$

$$\sum_{K=1}^{K} \pi_K = 0 \quad (\mathbb{1})$$

Therefore, $\sum_{K=1}^{K} Q_K = \underbrace{\sum_{K=1}^{K} \hat{Q}_K}_{=1} + \lambda B(\mathbb{1}) \underbrace{\sum_{K=1}^{K} \pi_K}_{=0} > 1$

and we can conclude that Q is a prob. measure.

Next

$$E_Q[S_n^*(l)] = \sum_{k=1}^{K} Q_k \frac{S_n(l,w_k)}{B(l)}$$

$$= \sum_{k=1}^{K} \hat{Q}_k \frac{S_n(l,w_k)}{B(l)} + \lambda \underbrace{\sum_{k=1}^{K} \pi_k S_n(l,w_k)}_{=0,\ \text{since } \pi^T S(l,\cdot)=0}$$

$$= \sum_{k=1}^{K} \hat{Q}_k S_n^*(l,w_k) = E_{\hat{Q}}[S_n^*(l)] = S_n^*(0) \Rightarrow Q \in M$$

Now, we only need to check that $E_Q[X/B(l)] \neq E_{\hat{Q}}[X/B(l)]$

$$E_Q[X/B(l)] = \sum_{k=1}^{K} \hat{Q}_k \frac{X(w_k)}{B(l)} + \lambda \sum_{k=1}^{K} \pi_k X(w_k)$$

$$= E_{\hat{Q}}[X/B(l)] + \delta, \text{ with } \delta \neq 0$$

because $\lambda \neq 0$ and $\pi^T X > 0$