Proof of Lemma 5.3

Suppose that $\hat{H}$ is an optimal solution of the portfolio problem and $\tilde{H}$ is an arbitrage opportunity. That is, $\tilde{V}(0) = 0$, $\tilde{V}(1) \geq 0$ and

$\tilde{V}(1, \tilde{\omega}) > 0$ for at least one $\tilde{\omega} \in \Omega$.

Define $H = \hat{H} + \tilde{H}$.

Then:

$$V(0) = \hat{V}(0) + \tilde{V}(0) = v + 0 = v$$

and

$$V^H(1) = v + \underbrace{\sum_{n=1}^{N} H_n \Delta S_n^*}_{G^H} = \underbrace{v + \sum_{n=1}^{N} \hat{H}_n \Delta S_n^*}_{\hat{V}^*(1)} + \underbrace{\sum_{n=1}^{N} \tilde{H} \Delta S_n^*}_{\tilde{V}^*(1)}$$

$\Rightarrow$ $V^*(1) \geq \hat{V}^*(1)$ and $V^*(1,\tilde{\omega}) > \hat{V}^*(1,\tilde{\omega})$ for at least one $\tilde{\omega} \in \Omega$.

$\uparrow$

$\tilde{V}^*(1) \geq 0$, $\tilde{V}(1,\tilde{\omega}) > 0$ for at least one $\tilde{\omega} \in \Omega$.

Since u is strictly increasing it follows that

$$E[u(V(1))] = E[u(B(1)V^*(1))]$$
$$> E[u(B(1)\hat{V}^*(1))]$$
$$= E[u(\hat{V}(1))]$$

contradicting the optimality of $\hat{H}$.

Proof of Lemma 55

The function to be optimized in O.P.P. is

$$F:\ (H_1,\dots H_N) \longmapsto \sum_{\omega\in\Omega} u\left(B(1,\omega)\left(v+\sum_{n=1}^{N} H_n\,\Delta S_n^*(\omega)\right),\omega\right)P(\omega)$$

The first order necessary condition is given by

$$\frac{\partial F}{\partial H_m}=0,\ m=1,\dots N \text{ or}$$

$$0=\sum_{\omega\in\Omega} u'\left(B(1,\omega)\left\{v+\sum_{n=1}^{N} H_n\,\Delta S_n^*(\omega)\right\},\omega\right)B(1,\omega)\,\Delta S_m^*(\omega)\,P(\omega) \qquad m=1,\dots N$$

$$=E\left[u'\left(V(1)\right)B(1)\,\Delta S_m^*\right]$$

(*)

Define

$$Q(\omega) = \frac{B(l,\omega)\, u'(V(l,\omega),\omega)}{E[\, B(l)\, u'(V(l))\,]}\, P(\omega),$$

then, since $B(l) > 0$ and $u' > 0$ (strictly increasing) we have that $Q(\omega) > 0$, $\forall \omega \in \Omega$.

$$\sum_{\omega \in \Omega} Q(\omega) = \frac{1}{E[\, B(l)\, u'(V(l))\,]} \sum_{\omega \in \Omega} B(l,\omega)\, u'(V(l,\omega),\omega)\, P(\omega) = 1$$

and

$$E_Q[\, \Delta S_n^* \,] = \sum_{\omega \in \Omega} \Delta S_n^*\, Q(\omega)$$

$$= \frac{1}{E[\, B(l)\, u'(V(l))\,]} \sum_{\omega \in \Omega} \Delta S_n^*\, B(l,\omega)\, u'(V(l,\omega),\omega)\, P(\omega)$$

$$= \frac{1}{E[B(L) M'(V(L))]} E[M'(V(L)) B(L) \Delta S_n^*] \overset{(*)}{=} 0$$

for all $n = 1, \dots, N$. Hence Q is a R.N.D.M.

Proof of Prop. 5.2

Lemma 5.5 proves $\Rightarrow$)

To prove $\Leftarrow$). Suppose $\exists Q \in M$. Define

$$M(n, w) = \frac{n}{B(1,w)} \frac{Q(w)}{P(w)}$$ (strictly increasing and differentiable and concave, for $w \in \Omega$.

For any bnd. strategy $H = (H_0, H_1, \dots, H_N)^T$ we have

$$E[M(B(1)(v + \sum_{n=1}^{N} H_n \Delta S_n^*))]$$

$$= \sum_{w \in \Omega} B(1,w)(v + \sum_{n=1}^{N} H_n \Delta S_n^*(w)) \frac{Q(w)}{B(1,w) P(w)} P(w)$$

$$= \sum_{\omega \in \Omega} \left(w + \sum_{n=1}^{N} H \cdot \Delta S_n^*(\omega) \right) Q(\omega)$$

$$= w + \sum_{n=1}^{N} H \cdot \underbrace{E_Q[\Delta S_n^*]}_{0} = w$$

Any H gives the same expected utility of the terminal wealth. Therefore, any trading strategy is optimal and we get utility u.