Example 62

Let $U(w) = -\exp(-w)$, $U'(w) = \exp(-w)$

Then $I(y) = -\log(y)$

The optimal wealth is given by

$$\hat{W} = -\log\left(\hat{\lambda} L B^{-1}(1)\right) = -\log(\hat{\lambda}) - \log(L B^{-1}(1))$$

The optimal Lagrange multiplier is given by

$$v = E_Q\left[\hat{W}/B(1)\right] = -E_Q[B^{-1}(1)]\log(\hat{\lambda}) - E_Q[B^{-1}(1)\log(L B^{-1}(1))]$$

$$\Rightarrow \hat{\lambda} = \exp\left\{-\frac{v - E_Q[B^{-1}(1)\log(L B^{-1}(1))]}{E_Q[B^{-1}(1)]}\right\}$$

Substituting $\hat{\lambda}$ in $\hat{w}$, we get

$$\hat{w} = \frac{r + E_Q[B^{-1}(1)\log(L B^{-1}(1))]}{E_Q[B^{-1}(1)]} - \log(L B^{-1}(1))$$

Computing $u(\hat{w})$ we get

$$u(\hat{w}) = -\exp\left\{-r + \frac{E_Q[B^{-1}(1)]\log(L B^{-1}(1) - E_Q[B^{-1}(1)\log(L B^{-1}(1)]}{E_Q[B^{-1}(1)]}\right\}$$

$$= -\hat{\lambda} L B^{-1}(1)$$

Therefore

$$E[u(\hat{w})] = -\hat{\lambda} E[L B^{-1}(1)] = -\hat{\lambda} E_Q[L^{-1} L B^{-1}(1)]$$

$$= -\hat{\lambda} E_Q[B^{-1}(1)]$$

If $B(0) = 1$, $B(1) = 1+r$, then $E[u(\hat{w})] = -\frac{\hat{\lambda}}{1+r}$

Suppose that $P = (1/2, 1/4, 1/4)^T$

For this market we know (or we can compute)

$Q = (1/3, 1/3, 1/3)^T$ and $r = 1/9$

Then, we have that

$$L = (2/3, 4/3, 4/3)^T \quad \text{and} \quad B(1) = \frac{10}{9}$$

In order to compute π we first compute

$$E_Q[\log(L B^{-1}(1))] = \frac{1}{3}\log\left(\frac{2/3}{10/2}\right) + \frac{1}{3}\log\left(\frac{4/3}{10/2}\right) + \frac{1}{3}\log\left(\frac{4/3}{10/2}\right)$$

$$= \frac{1}{3}\left[\log\left(\frac{3}{5}\right) + 2\log\left(\frac{6}{5}\right)\right]$$

$$= 1/3 \log\left(\frac{108}{125}\right)$$

This yields

$$\hat{\lambda} = \exp\left(\frac{-v - E_Q[B^{-1}(1) \log(L B^{-1}(1))]}{E_Q[B^{-1}(1)]} \right) = \exp\left\{ -\frac{10v}{9} - \frac{1}{3} \log\left(\frac{108}{125}\right) \right\}$$

and the optimal value of the expected utility

$$E[u(\hat{W})] = -\hat{\lambda} E_Q[B^{-1}(1)] = \frac{9}{10} \exp\left\{ -\frac{10v}{9} - \frac{1}{3} \log\left(\frac{108}{125}\right) \right\}$$

The optimal wealth is given by

$$\hat{W}_1 = -\log(\hat{\lambda}) - \log(L_1 B^{-1}(1,w_1)) = \frac{10}{9} v + \frac{1}{3} \log\left(\frac{108}{125}\right) - \log(3/5)$$

$$\hat{W}_2 = -\log(\hat{\lambda}) - \log(L_2 B^{-1}(1,w_2)) = \frac{10}{9} v + \frac{1}{3} \log\left(\frac{108}{125}\right) - \log(6/5)$$

$$\hat{W}_3 = -\log(\hat{\lambda}) - \log(L_3 B^{-1}(1,w_3)) = \frac{10}{9} v + \frac{1}{3} \log\left(\frac{108}{125}\right) - \log(6/5)$$

The discounted optimal wealth is given

$$\hat{W}_1^* = v + 3/10 \log\left(\frac{108}{125}\right) - \frac{9}{10} \log(3/5)$$

$$\hat{W}_2^* = v + 3/10 \log\left(\frac{108}{125}\right) - \frac{9}{10} \log(6/5)$$

$$\hat{W}_3^* = v + 3/10 \log\left(\frac{108}{125}\right) - \frac{9}{10} \log(6/5)$$

In general, $V^*(l) = v + G^* = v + H_1 \Delta S_1^* + H_2 \Delta S_2^*$

Since $\Delta S_1^* = (0, 2, -2)^T$ and $\Delta S_2^* = (3, -1, -2)^T$

we get the equations

$\hat{w}_1^* = \frac{3}{10} \log\left(\frac{108}{125}\right) - \frac{9}{10} \log(3/5) + \cancel{v} = \cancel{v} + H_1 \cdot 0 + H_2 \cdot 3$

$\hat{w}_2^* = \frac{3}{10} \log\left(\frac{108}{125}\right) - \frac{9}{10} \log(6/5) + \cancel{v} = \cancel{v} + H_1 \cdot 2 + H_2 (-1)$

$\hat{w}_3^* = \frac{3}{10} \log\left(\frac{108}{125}\right) - \frac{9}{10} \log(6/5) + \cancel{v} = \cancel{v} + H_1 (-2) + H_2 (-2)$

$\Rightarrow H_2 = \frac{1}{10} \log\left(\frac{108}{125}\right) - \frac{3}{10} \log(3/5) \simeq 0.153663$

$H_1 = \frac{1}{5} \log\left(\frac{108}{125}\right) - \frac{9}{20} \log(6/5) - \frac{3}{20} \log(3/5) \simeq -0.033466$

To find H_0, note that

$v = V(0) = H_0 + H_1 S_1(0) + H_2 S_2(0) = H_0 + 6 H_1 + 10 H_2$

$\Rightarrow H_0 = v - 1.117054$