Proof of Proposition 30

Clearly $E[X|\mathcal{F}]$ is $\mathcal{F}$-measurable because it is constant on the elements of the partition $\pi = \{A_1, \dots, A_m\}$

We only need to check condition 2. in Def. 27.

Any $B \in \mathcal{F}$ is of the form $B = A_{i_1} \cup \dots \cup A_{i_n}$, where $i_j \in \{1, \dots, m\}$ $j = 1, \dots, n$. Therefore,

- $E[X 1_B] = E\left[X \sum_{j=1}^{n} 1_{A_{i_j}}\right] = \sum_{j=1}^{n} E[X 1_{A_{i_j}}]$
- $E[E[X|\mathcal{F}] 1_B] = E\left[\left(\sum_{i=1}^{m} E[X|A_i] 1_{A_i}\right)\left(\sum_{j=1}^{n} 1_{A_{i_j}}\right)\right]$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} E[E[X|A_i] 1_{A_i \cap A_{i_j}}]$$

$$= \sum_{j=1}^{n} E[E[X|A_{i_j}] 1_{A_{i_j}}]$$

$$= \sum_{j=1}^{\infty} E\left[\frac{E[X 1_{A_{ij}}]}{P(A_{ij})} 1_{A_{ij}} \right] = \sum_{j=1}^{\infty} E[X 1_{A_{ij}}]$$