Example of Remark 2.4

$\Omega = \{\omega_1, \omega_2, \ldots, \omega_6\}$

$S = \{\underbrace{\{\omega_1\}}_{A_1}, \underbrace{\{\omega_1, \omega_4\}}_{A_2}, \underbrace{\{\omega_1, \omega_2, \omega_5\}}_{A_3}\}$

$\sigma(S) = ?$

$\left(\bigcup_{i=1}^{3} A_i\right)^c = \{\omega_3, \omega_6\} =: B_1$

$A_1^c \cap A_2 = \{\omega_4\} =: B_2$

$A_1^c \cap A_3 = \{\omega_2, \omega_5\} =: B_3$

$\Rightarrow \Pi = \{\underbrace{\{\omega_1\}}_{A_1}, \underbrace{\{\omega_2, \omega_5\}}_{B_3}, \underbrace{\{\omega_4\}}_{B_2}, \underbrace{\{\omega_3, \omega_6\}}_{B_1}\}$

$a(S) = a(\mathcal{T}) = \{\emptyset, \{w_2\}, \{w_2, w_7\}, \{w_1\}, \{w_3, w_6\},$
$A_2 \quad B_3 \quad B_2 \quad B_2$

$\{w_1, w_2, w_7\}, \{w_1, w_1\}, \{w_2, w_3, w_6\},$
$A_2 \cup B_3 \quad A_2 \cup B_2 \quad A_2 \cup B_2$

$\{w_1, w_2, w_7, w_4\}, \{w_2, w_7, w_4\},$
$A_2 \cup B_3 \cup B_2 \quad B_3 \cup B_2$

$\{w_2, w_7, w_3, w_6\}, \{w_2, w_7, w_4, w_3, w_6\}, \{w_2, w_3, w_6\}$
$B_3 \cup B_2 \quad B_3 \cup B_2 \cup B_2 \quad B_2 \cup B_2$

$\{w_2, w_2, w_7, w_3, w_6\}, \{w_2, w_4, w_3, w_6\}\}$
$A_2 \cup B_3 \cup B_2 \quad A_2 \cup B_2 \cup B_2$

Let $\mathcal{F} = a(\pi)$ where π is a partition of Ω

X is $\mathcal{F}$-measurable $\Leftrightarrow$ X is constant on each element of the partition

($\Leftarrow$) Let $\pi = \{A_1, \dots, A_m\}$ $m \leq K = \#\Omega$

$$X(\omega) = \sum_{j=1}^{m} a_j 1\!\!1_{A_j}(\omega)$$

X takes m different values and

$X^{-1}(a_j) = A_j \in \pi \Rightarrow X^{-1}(a_j) \in a(\pi) = \mathcal{F}$

$\Rightarrow$ X is $\mathcal{F}$-measurable

✗ takes $m' < m$ different values.

$X^{-1}(a_j)$ = finite union of A_j, but still $\in \mathcal{F}$.

$\Longrightarrow$ Suppose $\exists j_0 \in \{1,\dots,m\}$ and $w_1, w_2 \in A_{j_0}$

such that $X(w_1) = x_1 \neq x_2 = X(w_2)$

Since $X^{-1}(x_1) \cap X^{-1}(x_2) = \emptyset$ we get

that $X^{-1}(x_1) \cap A_{j_0} \subsetneq A_{j_0}$

By assumption X is $\mathcal{F}$-measurable $\Rightarrow X^{-1}(x_1) \in \mathcal{F}$

$\rightarrow X^{-1}(x_1) \cap A_{j_0} \in \mathcal{F}$ but we have just seen

that $X^{-1}(x_1) \cap A_{j_0}$ is a proper subset of A_{j_0}.

$\Rightarrow$ it cannot be $\mathcal{F}$-measurable.

Example 32

$\Omega = \{\omega_1, \dots, \omega_4\}$ $\quad P = (1/4, \dots 1/4)^T$

$\mathcal{F} = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$

$$X(\omega) = \begin{cases} 9 & \omega = \omega_1 \\ 6 & \omega = \omega_2, \omega_3 \\ 3 & \omega = \omega_4 \end{cases}$$

$$E[X \mid \mathcal{F}](\omega) = E[X \mid \{\omega_1, \omega_2\}]\, 1_{\{\omega_1, \omega_2\}}(\omega) + E[X \mid \{\omega_3, \omega_4\}]\, 1_{\{\omega_3, \omega_4\}}(\omega) = \frac{15}{2} 1_{\{\omega_1, \omega_2\}}(\omega) + \frac{9}{2} 1_{\{\omega_3, \omega_4\}}(\omega)$$

$$E[X \mid \{\omega_1, \omega_2\}] = \sum_{\omega \in \{\omega_1, \omega_2\}} X(\omega) \frac{P(\omega)}{P(\{\omega_1, \omega_2\})}$$

$$= 9 \frac{1/4}{1/2} + 6 \frac{1/4}{1/2} = \frac{15}{2}$$

$$E[X \mid \{\omega_3, \omega_4\}] = \sum_{\omega \in \{\omega_3, \omega_4\}} X(\omega) \frac{P(\omega)}{P(\{\omega_3, \omega_4\})}$$

$$= 6 \frac{1/4}{1/2} + 3 \frac{1/4}{1/2} = 9/2$$

Proof of Theorem 3) $\mathcal{G} = \sigma(A_1, \dots, A_n)$ A_j is a partition of Ω.

1) Clearly $a E[X|\mathcal{G}] + b E[Y|\mathcal{G}]$ is $\mathcal{G}$-measurable
(constant on the elements of the partition generating $\mathcal{G}$)

Let $A \in \mathcal{G}$

$$E[(aX + bY)1_A] \overset{\text{linearity of } E[\cdot]}{=} a E[X 1_A] + b E[Y 1_A]$$

(Def of $E[X|\mathcal{G}]$ and $E[Y|\mathcal{G}]$)
$$= a E[E[X|\mathcal{G}] 1_A] + b E[E[Y|\mathcal{G}] 1_A]$$

linearity of $E[\cdot]$
$$= E[(a E[X|\mathcal{G}] + b E[Y|\mathcal{G}]) 1_A]$$

Candidate satisfies property 2) of cond. exp.

2) $E[E[X|\mathcal{G}]] = E[E[X|\mathcal{G}] 1_\Omega]$

Def. of $E[X|\mathcal{G}]$ $= E[X 1_\Omega] = E[X]$

since Ω belongs to any algebra

3) $E[X]$ is a constant. Hence, it is measurable with respect to any algebra.

Let $A \in \mathcal{G}$ and assume that $X = 1\!\!1_B$, $B \in \mathcal{F}(\Omega)$, and independent of $\mathcal{G}$.

(Since any X can be written as a linear combination of indicators of events and the expectation is a linear operator, there is no loss of generality).

$$E[X\,1\!\!1_A] = E[1\!\!1_B\,1\!\!1_A] = E[1\!\!1_{A\cap B}]$$

$$= P(A\cap B) \overset{\uparrow}{=} P(A)\,P(B)$$

(↑ independence assumption)

$$= E[1\!\!1_A]\,E[1\!\!1_B]$$

$$= E[E[1\!\!1_B]\,1\!\!1_A]$$

$$= E[E[X]\,1\!\!1_A] \quad \checkmark$$

4) Y is $\mathcal{G}$-measurable and $E[X|\mathcal{G}]$ is $\mathcal{G}$-measurable

. The product of $\mathcal{G}$-measurable r.v. is constant over the elements of the partition generating $\mathcal{G}$. $\Rightarrow$ It is $\mathcal{G}$-measurable.

Let $A \in \mathcal{G}$ and $Y = 1_{A_i}$ where $A_i \in \pi$

$$E[X Y 1_A] = E[X 1_{A_i \cap A}]$$

$A \cap A_i \in \mathcal{G}$ $$= E[E[X|\mathcal{G}] 1_{A_i \cap A}]$$

def. of $E[X|\mathcal{G}]$ $$= E[1_{A_i} E[X|\mathcal{G}] 1_A]$$

$$= E[Y E[X|\mathcal{G}] 1_A]$$

the cndtl. satisfies property 2) of conditional expectation.

3) $\mathcal{H} \subseteq \mathcal{G}$ $\quad E[X|\mathcal{H}]$ is $\mathcal{H}$-measurable + $\mathcal{H} \subseteq \mathcal{G}$ $\Rightarrow$ $E[X|\mathcal{H}]$ is $\mathcal{G}$-measurable.

- $E[E[X|\mathcal{H}]|\mathcal{G}] \overset{?}{=} E[X|\mathcal{H}]$

  $\|$ $\Leftarrow$ 4) $E[X|\mathcal{H}]$ is is $\mathcal{G}$-measurable because $\mathcal{H} \subseteq \mathcal{G}$.

  $E[X|\mathcal{H}]\ E[2|\mathcal{G}]$

  $\|$ $\leftarrow$ Constants are independent of any algebra + 3)

  $E[X|\mathcal{H}]$

- $E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$

  The candidate is $\mathcal{H}$-measurable ✓

For any $A \in \mathcal{H}$

$$E[E[X|\mathcal{G}]\,1_A] = E[X\,1_A] = E[E[X|\mathcal{H}]\,1_A]$$

$A \in \mathcal{H} \subseteq \mathcal{G}$, def. of $E[\cdot|\mathcal{G}]$ ↑ def. of $E[\cdot|\mathcal{H}]$ ↑

↳ Candidate satisfies the condition 2) in the def. of cond. algebra

Proof of Theorem 34

Let $\pi = \{A_1, \dots, A_m\}$ be a partition of Ω such that $\mathcal{G} = \sigma(\pi)$

We recall that if Y is $\mathcal{G}$-measurable then Y is of the form

$$Y = \sum_{j=1}^{m} y_j 1\!\!1_{A_j}$$

Then,

$$J(Y) = E[(X-Y)^2] = E\left[\left(X - \sum_{j=1}^{m} y_j 1\!\!1_{A_j}\right)^2\right]$$

$$= E[X^2] - 2E\left[X \sum_{j=1}^{m} y_j 1\!\!1_{A_j}\right] + E\left[\left(\sum_{j=1}^{m} y_j 1\!\!1_{A_j}\right)^2\right]$$

linearity of $E[\cdot]$ $= E[X^2] - 2\sum_{j=1}^{m} y_j E[X 1\!\!1_{A_j}] + \sum_{j=1}^{m} y_j^2 P(A_j)$

$1\!\!1_{A_i} 1\!\!1_{A_j} = \begin{cases} 1\!\!1_{A_i} & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

We may think $J(Y)$ as a function of m variables $y_1, \dots, y_m$

$$0 = \frac{\partial J(\hat{y})}{\partial \hat{y}_j} = -2\,E[X\,1_{A_j}] + 2\,\hat{y}_j\,P(A_j) \qquad j = 1,\dots,m.$$

$$\Rightarrow \hat{y}_j = \frac{E[X\,1_{A_j}]}{P(A_j)} = E[X \mid A_j]$$

Hence,

$$\hat{y} = \sum_{j=1}^{m} E[X \mid A_j]\,1_{A_j} = E[X \mid \mathcal{G}]$$

Prop. 30