Example 1.2

$N = 2$, $K = 4$, $B(t) = (1+r)^t$, $r > 0$, $T = 2$

$S(0) = 5$, $S(1,\omega) = \begin{cases} 8 & \omega = \omega_1, \omega_2 \\ 4 & \omega = \omega_3, \omega_4 \end{cases}$, $S_2(2,\omega) = \begin{cases} 9 & \omega = \omega_1 \\ 6 & \omega = \omega_2, \omega_3 \\ 3 & \omega = \omega_4 \end{cases}$

$\mathcal{F}_0 = \{\Omega, \emptyset\}$, $\mathcal{F}_2 = \sigma(S(0), S(1)) = \sigma()(1)) = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$

$\mathcal{F}_2 = \mathcal{P}(\Omega)$.

Let $H = \{ H(t) = (H_0(t), H_2(t))^T \}_{t=1,2}$ be a trading strategy

$H(1)$ is $\mathcal{F}_0$-measurable and $H(2)$ is $\mathcal{F}_2$-measurable.

$H(1)$ is $\mathcal{F}_0$-measurable $\Rightarrow$ $H(1)$ is a vector of constants.

$$V(0) = H_0(1) \cdot 1 + S \cdot H_2(1)$$

$$V(1,\omega) = \begin{cases} (1+r) H_0(1) + 8 H_2(1) & \omega = \omega_1, \omega_2 \\ (1+r) H_0(1) + 4 H_2(1) & \omega = \omega_3, \omega_4 \end{cases}$$

$$V(z,w) = \begin{cases} (l+r)^2 H_0(z,w_1) + 9 H_2(z,w_1) & w = w_1 \\ (l+r)^2 H_0(z,w_2) + 6 H_2(z,w_2) & w = w_2 \\ (l+r)^2 H_0(z,w_3) + 6 H_2(z,w_3) & w = w_3 \\ (l+r)^2 H_0(z,w_4) + 3 H_2(z,w_4) & w = w_4 \end{cases}$$

$H(z)$ is $\mathcal{F}_l$-measurable $\Rightarrow$ $H_2(z)$ is constant over $\{w_1,w_2\}, \{w_3,w_4\}$

$\Rightarrow$ $H_0(z,w_1) = H_0(z,w_2) = a$
$H_2(z,w_1) = H_2(z,w_2) = b$
$H_0(z,w_3) = H_0(z,w_4) = c$
$H_2(z,w_3) = H_2(z,w_4) = d$

We compute $\Delta S(z,w) = \begin{cases} 8-5=3 & w = w_1, w_2 \\ 4-8=-4 & w = w_3, w_4 \end{cases}$

and $\Delta S(z,w) = \begin{cases} 9-8=1 & w = w_1 \\ 6-8=-2 & w = w_2 \\ 6-4=2 & w = w_3 \\ 3-4=-1 & w = w_4 \end{cases}$

$\Delta B(1) = r$ and $\Delta B(l) = (l+r)^2 - (l+r) = r(r+l)$

$$G(1) = H_0(1)\,\delta B(1) + H_1(1)\,\delta S(1) = \begin{cases} 2H_0 + 3H_1 & \omega = \omega_1, \omega_2 \\ 2H_0 - H_1 & \omega = \omega_3, \omega_4 \end{cases}$$

$$G(2) = G(1) + H_0(2)\,\delta B(2) + H_1(2)\,\delta S(2)$$

$$= \begin{cases} 2H_0 + 3H_1 + 2(2+1)\,a + b & \omega = \omega_1 \\ 2H_0 + 3H_1 + 2(2+1)\,a - 2b & \omega = \omega_2 \\ 2H_0 - H_1 + 2(2+1)\,c + 2d & \omega = \omega_3 \\ 2H_0 - H_1 + 2(2+1)\,c - d & \omega = \omega_4 \end{cases}$$

For H to be self-financing we must have

$$V(1,\omega) = (1+2)\,H_0(1) + \$\,H_1(1) = (1+2)\,H_0(2,\omega) + \$\,H_1(2,\omega) \quad \omega = \omega_1, \omega_2$$

$$V(1,\omega) = (1+2)\,H(1) + 4\,H_1(1) = (1+2)\,H_0(2,\omega) + 4\,H_1(2,\omega) \quad \omega = \omega_3, \omega_4$$

Example 15 (continuation of the previous example)

We will fix $Q = (Q_1, \dots, Q_4)^T$ satisfying

$$E_Q[S(t+1) \mid \mathcal{F}_t] = (1+r) S(t) \qquad t = 0, 1$$

$t = 0$

$\mathcal{F}_0 = \{\emptyset, \Omega\} = \sigma\{\{\Omega\}\}$

$(1+r) S(0) = E_Q[S(1) \mid \mathcal{F}_0] = E_Q[S(1)]$

$\Updownarrow$

$$8(Q_1 + Q_2) + 4(Q_3 + Q_4) = (1+r) S \qquad (*)$$

$t = 1$

$\mathcal{F}_1 = \sigma(\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}) =: \sigma(\{A_1, A_2\}) = \{\emptyset, \Omega, A_1, A_2\}$

$(1+r) S(1) = E_Q[S(2) \mid \mathcal{F}_1] = E_Q[S(2) \mid A_1] \mathbb{1}_{A_1} + E_Q[S(2) \mid A_2] \mathbb{1}_{A_2}$

- $E_Q[S(2) \mid A_2] = S(2,\omega_1)\frac{Q(\omega_1)}{Q(A_2)} + S(2,\omega_2)\frac{Q(\omega_2)}{Q(A_2)}$

$$= 9\,\frac{Q_1}{Q_1+Q_2} + 6\,\frac{Q_2}{Q_1+Q_2}$$

- $E_Q[S(2) \mid A_2] = S(2,\omega_3)\frac{Q(\omega_3)}{Q(A_2)} + S(2,\omega_4)\frac{Q(\omega_4)}{Q(A_2)}$

$$= 6\,\frac{Q_3}{Q_3+Q_4} + 3\,\frac{Q_4}{Q_3+Q_4}$$

Note that $S(A\omega) = S(2,A_2)\,1\!\!1_{A_2}(\omega) + S(1,A_2)\,1\!\!1_{A_2^c}(\omega) = 8\,1\!\!1_A(\omega) + 4\,1\!\!1_{A^c}(\omega)$

Hence, we get

$$9Q_1 + 6Q_2 = 8(1+2)(Q_1+Q_2) \quad (**)$$
$$6Q_3 + 3Q_4 = 4(1+2)(Q_3+Q_4) \quad (***)$$

Since Q must be a probability $\quad Q_1+Q_2+Q_3+Q_4 = 1$

We have 4 variables and 4 equations and we obtain

$$Q_1 = \frac{1+5n}{4} \cdot \frac{2+8n}{3}, \quad Q_3 = \frac{3-5n}{4} \frac{1+4n}{3}$$

$$Q_2 = \frac{1+5n}{4} \frac{1-8n}{3}, \quad Q_4 = \frac{3-5n}{4} \cdot \frac{2-4n}{3}$$

Moreover, $Q > 0 \Rightarrow 0 \leq n < 1/8$

Proof of Proposition 17

Define $G^{*,n} = \{ G^{*,n}(t) = \sum_{u=1}^{t} H_n(u) \Delta S_n^*(u) \}_{t=0,1,\dots,T}$

$\{ G^{*,n}(0) = 0 \}$ $n = 1, \dots, N$.

By definition (of G) S_n^*, $n = 1, \dots, N$ are martingales under Q

Moreover, since H is predictable, by Proposition 38 we get that $G^{*,n}$ are also martingales under Q.

Note that a constant is always a martingale and by the linearity of the conditional expectation the set of martingales is a linear vector space.

Hence, since H is self-financing by equation (2)

$$V^*(t) = V^*(0) + G^*(t) = V^*(0) + \sum_{n=1}^{N} G^{*,n}(t)$$

is a martingale under Q.