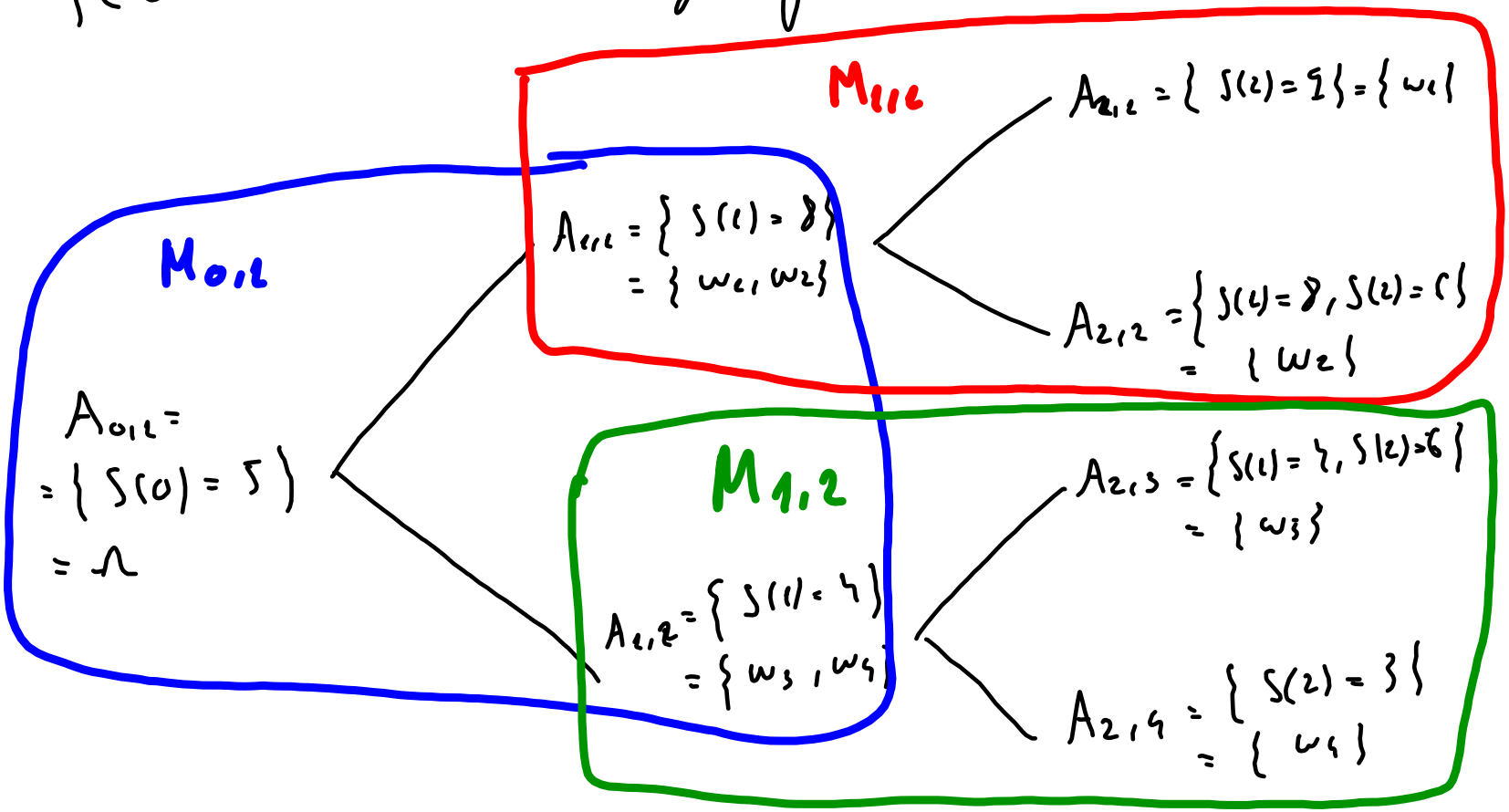
Example 17 (Alternative method)

Reduction to single period markets



There are 3 single period markets embedded $M_{0,1}, M_{1,1}, M_{1,2}$

Recall that $P = (1/4, 1/4, 2/4, 1/4)^T$.

$M_{0,1}$

$\Omega_{0,1} = \{A_{1,1}, A_{1,2}\}$, $P_{0,1} = P(0, A_{0,1}) = (P(A_{1,1} \mid A_{0,1}), P(A_{1,2} \mid A_{0,1}))^T$

$= \left(\frac{P(A_{1,1} \cap A_{0,1})}{P(A_{0,1})}, \frac{P(A_{1,2} \cap A_{0,1})}{P(A_{0,1})} \right)^T = \left(\frac{1/2}{1}, \frac{1/2}{1} \right)^T = (1/2, 1/2)^T$

The RNPM for $M_{0,1}$ is $Q_{0,1} = Q(0, A_{0,1}) = (Q(A_{1,1} \mid A_{0,1}), Q(A_{1,2} \mid A_{0,1}))^T$

Satisfying

$E_{Q_{0,1}}[S^*(1)] = S^*(0) \Leftrightarrow E_{Q(0, A_{0,1})}[S(1)] = (1+r) S(0)$

$\Leftrightarrow \begin{cases} 8\, Q(A_{1,1} \mid A_{0,1}) + 4\, Q(A_{1,2} \mid A_{0,1}) = 5(1+r) \\ Q(A_{1,1} \mid A_{0,1}) + Q(A_{1,2} \mid A_{0,1}) = 1 \end{cases}$

$\Rightarrow Q(A_{1,1} \mid A_{0,1}) = \frac{1+5r}{4}$, $Q(A_{1,2} \mid A_{0,1}) = \frac{3-5r}{4}$

$M_{1,1}$

$\Omega_{1,1} = \{A_{2,1}, A_{2,2}\}$. $P_{1,1} = P(1, A_{1,1}) = (P(A_{2,1}/A_{1,1}), P(A_{2,2}/A_{1,1}))^T = (1/2, 1/2)^T$

The N.N.P.M. for $M_{1,1}$ is

$Q_{1,1} = Q(1, A_{1,1}) = (Q(A_{2,1}/A_{1,1}), Q(A_{2,2}/A_{1,1}))^T$ satisfying

$$E_{Q_{1,1}}[S^*(2)] = S^*(1) \Leftrightarrow$$

$$\Leftrightarrow E_{Q(1,A_{1,1})}\left[\frac{S(2)}{(1+2)^2}\right] = \frac{S(1)}{(1+2)}$$

$$\Leftrightarrow \begin{cases} q\,Q(A_{2,1}/A_{1,1}) + 6\,Q(A_{2,2}/A_{1,1}) = 8(1+2) \\ Q(A_{2,1}/A_{1,1}) + Q(A_{2,2}/A_{1,1}) = 1 \end{cases}$$

$$\Rightarrow Q(A_{2,1}/A_{1,1}) = \frac{2+\delta_2}{3}, \quad Q(A_{2,2}/A_{1,1}) = \frac{1-\delta_2}{3}$$

$M_{1:2}$

$\Omega_{1:2} = \{ A_{2:3}, A_{2:4} \}$, $P_{1:2} = P(L, A_{1:2}) = (P(A_{2:3} | A_{1:2}), P(A_{2:4} | A_{1:2}))^T$

$= (1/2, 1/2)^T$

The RMPA for $M_{1:2}$ is

$Q_{1:2} = Q(L, A_{1:2}) = (Q(A_{2:3} | A_{1:2}), Q(A_{2:4} | A_{1:2}))^T$

satisfying

$E_{Q_{1:2}}[S^*(2)] = S^*(1) \Leftrightarrow E_{Q(L, A_{1:2})}[S(2)] = (L+2) S(L)$

$\Leftrightarrow$ $\begin{cases} 6\, Q(A_{2:3} | A_{1:2}) + 3\, Q(A_{2:4} | A_{1:2}) = 4(L+2) \\ Q(A_{2:3} | A_{1:2}) + Q(A_{2:4} | A_{1:2}) = 1 \end{cases}$

$\Rightarrow Q(A_{2:3} | A_{1:2}) = \frac{L+2}{3}$, $Q(A_{2:4} | A_{1:2}) = \frac{2-L}{3}$

From the conditional risk neutral probabilities given $A_{0,1}$, $A_{1,1}$, $A_{1,2}$ we construct a RNPM for the multiperiod market.

$$Q(\omega_1) := Q(A_{1,1} \mid A_{0,1})\, Q(A_{2,2} \mid A_{1,2}) = \frac{1+\sqrt{2}}{4} \cdot \frac{2+\delta_2}{3}$$

$$Q(\omega_2) := Q(A_{1,2} \mid A_{0,1})\, Q(A_{2,2} \mid A_{1,1}) = \frac{1+\sqrt{2}}{4} \cdot \frac{1-\delta_2}{3}$$

$$Q(\omega_3) := Q(A_{1,2} \mid A_{0,1})\, Q(A_{2,3} \mid A_{1,2}) = \frac{3-\sqrt{2}}{4} \cdot \frac{1+\delta_2}{3}$$

$$Q(\omega_4) := Q(A_{1,2} \mid A_{0,1})\, Q(A_{2,3} \mid A_{1,2}) = \frac{3-\sqrt{2}}{4} \cdot \frac{2-\delta_2}{3}$$

Proof of Theorem 18

~~$\exists$ A.O.~~ $\iff \exists\, Q$ a martingale measure.

$(\Longleftarrow)$ Suppose H is self-financing, with $V^*(T) \geq 0$ and $E[V^*(T)] > 0$

Then $E_Q[V^*(T)] > 0$ (P and Q give strictly positive weight to each ω)

By Proposition 17, V^* is a martingale under Q and $P \sim Q$

$$V^*(0) = E_Q[V^*(0)] = E_Q[E_Q[V^*(T)|\mathcal{F}_0]]$$
$$= E_Q[V^*(T)] > 0$$

Hence, H cannot be an A.O., since H is arbitrary we can conclude.

$\Rightarrow$) If the multiple period market model is arbitrage free then all the underlying single period markets are also arbitrage free.

To see this, suppose that $\hat{H}$ is an A.O. for the single period market associated to $A \in \mathcal{F}_t$ for $t < T$ ($\mathcal{F}_t = a(\pi_t)$)

This means that $\hat{H} \Delta S_1^*(t+1) + \cdots + \hat{H} \Delta S_N^*(t+1) \geq 0$ and not identical to 0 on A.

Then, the following strategy is an A.O. in the multiperiod market

$$H_m(s,w) = \begin{cases} \hat{H}_m & s = t+1, w \in A, m = 1, \dots, N \\ -\hat{H}_1 S_1^*(t) - \cdots - \hat{H}_N S_N^*(t) & s = t+1, w \in A, m = 0 \\ \hat{H}_1 \Delta S_1^*(s+1) + \cdots + \hat{H}_N S_N^*(s+1) & s > t+1, w A, m = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, we can find a R.N.P.M. for each underlying single period market (as in example 15 (alternative method))

The martingale measure for the multiperiod market is constructed by "pasting together" these R.N.P.M.

We have $\mathcal{F}_t = \sigma(\pi_t) = \sigma(\{A_{t,1}, A_{t,2}, \dots, A_{t,m_t}\})$ $m_t = \#\pi_t$ $t = 0, \dots, T$

For $i = 1, \dots, K$ there $\exists!\ (j_0(i), j_1(i), \dots, j_T(i)) \in \{1, \dots, m_0\} \times \dots \times \{1, \dots, m_T\}$ such that

$$\omega_i = A_{0,j_0(i)} \cap A_{1,j_1(i)} \cap \dots \cap A_{T,j_T(i)} \text{ and } A_{t+1,j_{t+1}(i)} \subseteq A_{t,j_t(i)} \quad t = 0, \dots, T-1.$$

Then,

$$Q(\omega_i) = \prod_{t=0}^{T-1} Q(A_{t,j_t(i)}) = \prod_{t=0}^{T-1} Q(A_{t+1,j_{t+1}(i)} \mid A_{t,j_t(i)})$$

This $Q > 0$ and $\sum_{i=1}^{K} Q(\omega_i) = 1$. Measure for arbitrary $1 < T$

and $m \in \{1, \dots, N\}$

$$E_Q[\Delta S^*_m(t+1) \mid \mathcal{F}_t] = \sum_{i=1}^{K} E_Q[\Delta S^*_m \mid A_{t,i}]\, 1_{A_{t,i}} = 0$$

(*) $E_Q[\Delta S^*_m(t+1) \mid A_{t,i}] = \dfrac{E_Q[\Delta S^*_m(t+1)\, 1_{A_{t,i}}]}{Q(A_{t,i})} = E_{Q(\cdot \mid A_{t,i})}[\Delta S^*_m(t+1)] = 0$