Proof of Theorem 27 (Sketch)

If $P_*(t) > V(t)$ or $P_X(t) < V(t)$

one can think of strategies given rise positive riskless profits (similar to the discussion for single period markets)

However, since $\mathcal{M} \neq \emptyset$, LOP holds and given two trading strategies replicating X, their values at time $t < T$ must coincide.

Finally, the equation for $P_X(t)$ follows from the fact that $V^*(t)$ is a martingale under Q.

Example 28 (continuation of Example 25)

$T = 2$, $K = 4$, $r = 0$

$S(0) = 5$, $S(1) = \begin{cases} 8 & \omega = \omega_1, \omega_2 \\ 4 & \omega = \omega_3, \omega_4 \end{cases}$, $S(2) = \begin{cases} 9 & \omega = \omega_1 \\ 6 & \omega = \omega_2, \omega_3 \\ 3 & \omega = \omega_4 \end{cases}$

$Q = (1/6, 1/12, 1/4, 1/2)^T$ is the unique martingale measure for this market.

a) European call option

$X = (4, 2, 2, 0)^T$

$V(2) = X$, $V(0) = E_Q\left[\frac{D(0)}{D(2)} X\right] = 1/6 \cdot 4 + \frac{1}{12} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{2} \cdot 0$

$\frac{D(0)}{D(2)} = 1$

$= 1$

recall $\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\})$

$V(t)$ and $V^*(t)$ are $\mathcal{F}_t$-measurable rand. variables

$$V(t) = B(t) \cdot V^*(t) = B(t)\, E_Q[\, V^*(2) \,|\, \mathcal{F}_t]$$

$$= B(t)\, E_Q\left[\, \frac{X}{B(2)} \,|\, \mathcal{F}_t\right]$$

In this case $B(t) = B(2|=t \Rightarrow \quad V(t) = E_Q[\, X | \mathcal{F}_t].$

Moreover.

$$E_Q[\, X \,|\, \mathcal{F}_t](\omega) = E_Q[\, X \,|\, \{\omega_1, \omega_2\}]\, 1_{\{\omega_1,\omega_2\}}(\omega)$$

$$+ E_Q[\, X \,|\, \{\omega_3, \omega_4\}]\, 1_{\{\omega_3,\omega_4\}}(\omega)$$

$$E_Q[\, X \,|\, \{\omega_1, \omega_2\}] = \frac{E_Q[\, X\, 1_{\{\omega_1,\omega_2\}}]}{Q(\{\omega_1,\omega_2\})} = \frac{4 \cdot 1/6 + 1 \cdot 1/12 + 0 \cdot 1/4 + 6 \cdot 1/2}{1/6 + 1/12}$$

$$= \frac{9/12}{3/12} = 3$$

$$E_Q[\, X \,|\, \{\omega_3, \omega_4\}] = \frac{E_Q[\, X\, 1_{\{\omega_3,\omega_4\}}]}{Q(\{\omega_3,\omega_4\})} = \frac{0 \cdot 1/6 + 0 \cdot 1/12 + 1 \cdot 1/4 + 0 \cdot 1/2}{1/4 + 1/2}$$

$$= 1/3$$

Find method to compute H replicating X.

- We must know the value process V for X ✓
- We solve

(*) $V(t) = H_0(t) B(t) + \sum_{n=1}^{N} H_n(t) S_n(t) \quad t = 1, 2$

having into account that H is predictable (**)

$t = 2$

$$\left.\begin{aligned} V(2,\omega_1) = 4 &= H_0(2,\omega_1)\cdot 1 + H_1(2,\omega_1)\cdot 9 \\ V(2,\omega_2) = 1 &= H_0(2,\omega_2)\cdot 1 + H_1(2,\omega_2)\cdot 6 \\ V(2,\omega_3) = 1 &= H_0(2,\omega_3)\cdot 1 + H_1(2,\omega_3)\cdot 6 \\ V(2,\omega_4) = 0 &= H_0(2,\omega_4)\cdot 1 + H_1(2,\omega_4)\cdot 3 \end{aligned}\right\} \quad (*)$$

$$\left.\begin{aligned} H_0(2,\omega_1) = H_0(2,\omega_2), \; H_0(2,\omega_3) = H_0(2,\omega_4) \\ H_1(2,\omega_1) = H_1(2,\omega_2), \; H_1(2,\omega_3) = H_1(2,\omega_4) \end{aligned}\right\} \quad (**)$$

This yields

$$H_0(2,w) = \begin{cases} -5 & w \in \{w_1, w_2\} \\ -1 & w \in \{w_3, w_4\} \end{cases}, \quad H_1(2,w) = \begin{cases} 1 & w \in \{w_1, w_2\} \\ 1/3 & w \in \{w_3, w_4\} \end{cases}$$

$t-1$

$$\left.\begin{aligned} V(1,w) = 3 &= H_0(1,w)\cdot 1 + H_1(1,w)\cdot 8 && w = w_1, w_2 \\ V(1,w) = 1/3 &= H_0(1,w)\cdot 1 + H_1(1,w)\cdot 4 && w = w_3, w_4 \end{aligned}\right\} (*)$$

$$H_0(1,w_1) = H_0(1,w_2) = H_0(1,w_3) = H_0(1,w_4) =: H_0(1)$$
$$H_1(1,w_1) = H_1(1,w_2) = H_1(1,w_3) = H_1(1,w_4) =: H_1(1)$$

This yields

$$H_0(1) = -7/3 \quad \text{and} \quad H_1(1) = 2/3$$

Second method to compute H

- The first step is the same as in the first method.

  $\Rightarrow$ $H_0(2)$, $H_2(2)$

- The second step is to compute $V(2)$

$$V(2) = \begin{cases} H_0(2,\omega)\, B(2) + H_2(2,\omega)\, S(2,\omega) = -5\cdot 2 + 2\cdot 8 = 3, & \omega = \omega_1, \omega_2 \\ H_0(2,\omega)\, B(2) + H_2(2,\omega)\, S(2,\omega) = -2\cdot 2 + 1/3 \cdot 4 = 1/3, & \omega = \omega_3, \omega_4 \end{cases}$$

- The third step is the same as in the first method ($k=2$)

  $\Rightarrow$ $H_0(2)$, $H_2(2)$

- Finally, $V(0) = H_0(2)\, B(2) + H_2(2)\, S(0)$

$$= -\frac{7}{3}\cdot 2 + \frac{2}{3}\cdot 5 = 2$$

Third method to compute H

We will replicate $Y = (4/3, 4/3, 0, 0)^T$

Recall that $\Delta S^*(2) = (2, -2, 2, -2)^T$

$\Delta S^*(1) = (3, 3, -2, -2)^T$

$$V^*(t-1) + \sum_{m=1}^{\tilde{N}} H_m(t)\, \Delta S^*_m(t) = V^*(t)$$

$$A_1 = \{w_1, w_2\} \;,\; A_2 = \{w_3, w_4\}$$

$t = 2$

$$V^*(1, A_1) + H_1(2, A_1) \cdot 1 = 4/3 \quad \Big\} \; w = w_1$$

$$V^*(1, A_1) + H_1(2, A_1) \cdot (-2) = 4/3 \quad \Big\} \; w = w_2$$

$$\Rightarrow \quad V^*(1, A_1) = 2$$

$$H_1(1, A_1) = 1/3$$

$$V^*(1, A_2) + H_1(2, A_2) \cdot (2) = 0 \quad \Big\} \; w = w_3$$

$$V^*(1, A_2) + H_1(2, A_2) \cdot (-1) = 0 \quad \Big\} \; w = w_4$$

$$\Rightarrow \quad V^*(1, A_2) = 0$$

$$H_1(1, A_2) = 0$$

$t=2$

$$V^*(0) + H_2(1) * 3 = 2 = V^*(1,\omega) \quad \omega \in A_1$$

$$V^*(0) + H_2(1) \times (-1) = 0 = V^*(1,\omega) \quad \omega \in A_2$$

$$\Rightarrow \quad V^*(0) = 1/2 \quad \text{and} \quad H_2(1) = 1/2$$

To compute H_0 we use

$$H_0(1) = V^*(0) - H_2(1)\, S(0) = \frac{1}{2} - \frac{1}{2} + 5 = -2$$

$$H_0(2) = V^*(1) - H_2(2)\, S(1)$$

$$= \begin{cases} 2 - 1/3 * 8 = -2/3 & \omega = \omega_1, \omega_2 \\ 0 - 0 \times 4 = 0 & \omega = \omega_3, \omega_4 \end{cases}$$

Note that $V^*(0) = 1/2$ is the same value obtained using the risk neutral pricing formula $1/2 = E_Q[\, Y/B(1)\,]$.