Proof of Prop. 35

Let t be an arbitrary time, n an arbitrary risky asset and A an arbitrary element in π_{t-1}, the partition generating $\mathcal{F}_{t-1}$.

Note that $H_n(t)\,1_A$ is the position in the asset n carried forward from $t-1$ to t when A occurs.

The first optimality condition corresponding to $H_n(t)\,1_A$ is:

$$0 = \sum_{\omega \in A} \mu'\left(B(T,\omega)\left\{v + G^*(T,\omega)\right\}\right) \Delta S_n^*(t,\omega)\, B(T,\omega)\, P(\omega)$$

$$= E\left[\mu'\left(B(T)\left\{v + G^*(T)\right\}\right) \Delta S_n^*(t)\, B(T)\, 1_A\right]$$

$$= E\left[\mu'\left(B(T)\,|\,v + G^*(T)\right\}\right) \Delta S_n^*(t)\, B(T)\,|\,A\right] P(A)$$

$$> 0$$

Since this happens for all $A \in \pi_{t-1}$ we get

$$E\left[B(t)\, u'(B(T)\, l_T + G^*(T))\, \Delta S_n^*(t) \mid \mathcal{F}_{t-1} \right] = 0$$

and if Q is defined as in the statement of the proposition, using exercise 5 b) of the last list with $L(T,\omega) = \dfrac{B(T,\omega)\, u'(V(T,\omega),\omega)}{E\left[B(T)\, u'(V(T)) \right]}$

$$E_Q\left[\Delta S_n^*(t) \mid \mathcal{F}_{t-1} \right] = 0$$

Example 36

$T = 2$, $K = 4$

$p = (1/4, 1/4, 1/4, 1/4)^T$, $0 \le r < 1/8$, $S(c) = 5$. $S(\omega) = \begin{cases} 8 & \omega = \omega_1, \omega_2 \\ 4 & \omega = \omega_3, \omega_4 \end{cases}$

$$S(2,\omega) = \begin{cases} 9 & \omega = \omega_1 \\ 6 & \omega = \omega_2, \omega_3 \\ 3 & \omega = \omega_4 \end{cases}$$

We know that the unique multiple means

$$Q = \left(\frac{(1+5r)(2+8r)}{12}, \frac{(1+5r)(2-8r)}{12}, \frac{(3-5r)(1+4r)}{12}, \frac{(3-5r)(2-4r)}{12} \right)^T$$

We want to solve the optimal portfolio problem with

$u(w) = \log(w) \Rightarrow u'(w) = \frac{1}{w} \Rightarrow I(y) = (u')^{-1}(y) = \frac{1}{y}$

$$L = \frac{Q}{p} = \left(\frac{(1+5r)(2+8r)}{3}, \frac{(1+5r)(2-8r)}{3}, \dots \right)$$

Next, we find $\hat{W}$

$$\hat{W} = I\left(\frac{\hat{\lambda} L}{B(2)}\right) = \frac{B(2)}{\hat{\lambda} L}$$

and the optimal multiplier $\hat{\lambda}$ (which is a constant)

$$E_Q\left[\frac{\hat{W}}{B(2)}\right] = v \iff E_Q\left[\frac{B(2)}{\hat{\lambda} L B(2)}\right] = v$$

$$\iff \hat{\lambda} = \frac{E_Q[L^{-1}]}{v} = \frac{1}{v}$$

$$E_Q[L^{-1}] = E[L L^{-1}] = 1$$

$$\left(E_Q[X] = \sum_{k=1}^{K} X(\omega_k) Q(\omega_k) = \sum_{k=1}^{K} X(\omega_k) L(\omega_k) P(\omega_k) = E_P[XL]\right)$$

$$L = \frac{Q}{P}$$

Therefore,

$$\hat{W} = v B(2) L^{-1}$$

and the optimal expected utility.

$$E[U(\hat{W})] = \log(v) + \log(B(2)) - E[\log(L)]$$

$$= \log(v(1+r)^2) - E[\log(L)]$$

$$= \log(v(1+r)^2) - \frac{1}{4}\sum_{i=1}^{4} \log(L_i)$$

The last step is to compute the optimal strategy $\hat{H}$ that replicates $\hat{W}$.

$A_1 = \{w_1, w_2\}$, $A_2 = \{w_3, w_4\}$

$t = 2$

$$\frac{3v(L+2)^2}{(L+5-2)(2+8-2)} = \hat{W}(w_1) = H_0(2,w)\underbrace{(L+2)^2}_{B(2)} + H_1(2,w) \times \frac{9}{S(2,w_1)} \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} w = w_1, w_2$$

$$\frac{3v(L+2)^2}{(L+5-2)(L-8-2)} = \hat{W}(w_2) = H_0(2,w)(L+2)^2 + H_1(2,w) \times \frac{6}{S(2,w_2)}$$

$$\Rightarrow H_0(2,w) = \frac{12(L+10-2)v}{(L+5-2)(L-8-2)(2+8-2)}, \quad H_1(2,w) = -\frac{(L+2)^2(L+10-2)v}{(L+5-2)(L-8-2)(L+8-2)}$$

for $w \in \{w_1, w_2\}$

Moreover we can compute (because A is self-financing)

$$V(L,w) = H_0(2,w)B(L) + H_1(2,w)S(L,w)$$

$$= \frac{2v(L+2)}{L+5-2} \qquad w \in \{w_1, w_2\}$$

For $w \in \{w_3, w_4\} = A_2$

$$\frac{3\nu(l+2)^2}{(3-5\nu)(l+4\nu)} = \widehat{W}(w_3) = H_0(2,w)(l+2)^2 + H_2(2,w) \times \frac{6}{S(2,w_3)}$$

$$\frac{3\nu(l+2)^2}{(3-5\nu)(2-4\nu)} = \widehat{W}(w_4) = H_0(2,w)(l+2)^2 + H_2(2,w) \times \frac{3}{S(2,w_4)}$$

$\Big\} \; w \in A_2$ (A_2 ↓)

$$\Rightarrow H_0(2,w) = \frac{3(2\nu}{(3-5\nu)(2-4\nu)(l+4\nu}, \quad H_2(2,w) = \frac{(l+2)^2(l-0\nu)\nu}{2(3-5\nu)(2-4\nu)(l+4\nu)}$$

for $w \in A_2$. Moreover

$$V(l,w) = H_0(2,w)B(l) + H_2(2,w)S(l,w)$$

$$= \frac{2\nu(l+2)}{3-5\nu} \qquad w \in \in A_2$$

$t = 2$

$$\frac{2v(2+n)}{2+5n} = V(2,w) = H_0(2)(2+n) + H_2(2) + \underbrace{\frac{1}{S(2,w)}}_{} \quad w \in A_2$$

$$\frac{2v(2+n)}{3-5n} = V(2,w) = H_0(2)(2+n) + H_2(2) + \frac{1}{S(2,w)} \quad w \in A_2$$

$$\Rightarrow H_0(2) = \frac{(30n-2)v}{(2+5n)(3-5n)}, \quad H_2(2) = \frac{(2+n)(2-5n)v}{(2+5n)(3-5n)}$$

To double check

$$V(2) = H_0(2) + H_2(2) + \frac{5}{S(2)} = v\,\frac{30n-2+(2+n)(2-5n)+5}{(2+5n)(3-5n)}$$

$$= v$$