Time value of money. Streams of Payments

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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Cash Flow

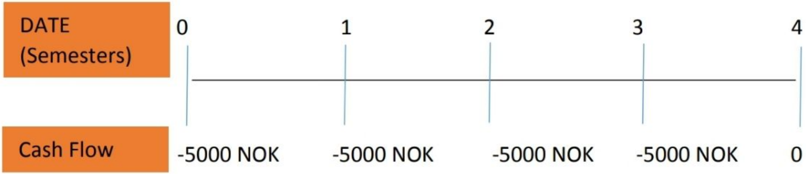
One of the main tools of financial analysis is the assessment of ***cash flow***, which is generated during a certain set of time intervals due to the implementation of a certain project or the operation of a particular type of asset. Revenues generated within a time interval are generally considered to occur at the beginning or end of the interval, that is, they are not spread out in the middle.

Cash Flow

Problem

Suppose you must pay tuition of 10,000 NOK per year for the next two years. Your tuition payments must be made in equal installments at the start of each semester. What is the timeline of your tuition payments?

Solution Assuming today is the start of the first semester, your first payment occurs at date 0 (today). The remaining payments occur at semester intervals. Using one semester as the period length, we can construct a timeline as follows:



Cash Flow

The Three Rules of Time Travel

Rule 1: Comparing and Combining Values

Our first rule is that it is only possible to compare or combine values at the same point in time.

Rule 2: Moving Cash Flows Forward in Time

To take a cash flow C forward n periods into the future, we must compound it by the n intervening interest rate factors.

$$\text{Future Value} = FV = C \cdot (1+r) \cdot (1+r) \cdot \ldots \cdot (1+r) = C \cdot (1+r)^n.$$

Rule 3: Moving Cash Flows Back in Time

To move a cash flow C backward n periods, we must discount it by the n intervening interest rate factors.

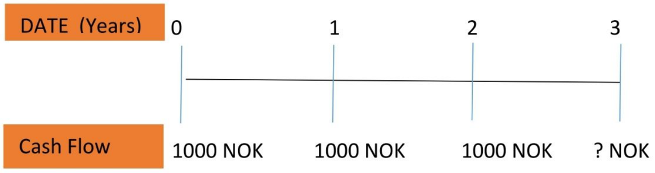
$$V(0) = \text{Present Value} = PV = C \cdot \frac{1}{(1+r)} \frac{1}{(1+r)} \cdot \ldots \cdot \frac{1}{(1+r)} = C \cdot \frac{1}{(1+r)^n}.$$

Cash Flow

Example

Suppose we plan to save 1000 NOK today, and 1000 NOK at the end of each of the next two years. If we earn a fixed 10 % interest rate on our savings, how much will we have three years from today?

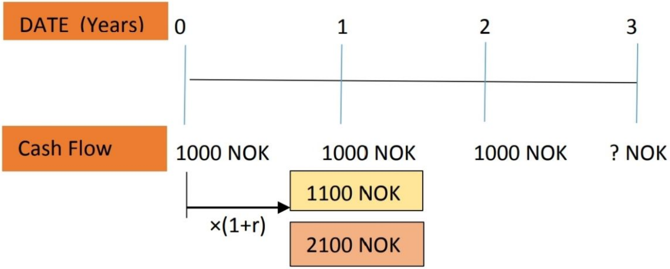
Solution. Timeline will the next:



The timeline shows the three deposits we plan to make. We need to compute their value at the end of three years.

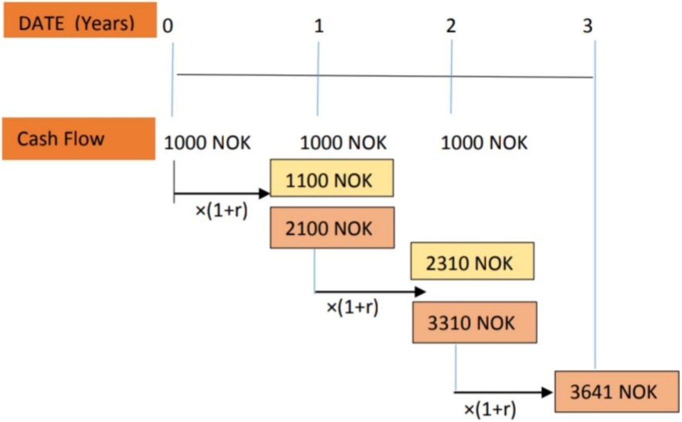
Cash Flow

We can use the rules of time travel in a number of ways to solve this problem. First, we can take the deposit at date 0 and move it forward to date 1. Because it is then in the same time period as the date 1 deposit, we can combine the two amounts to find out the total in the bank on date 1:



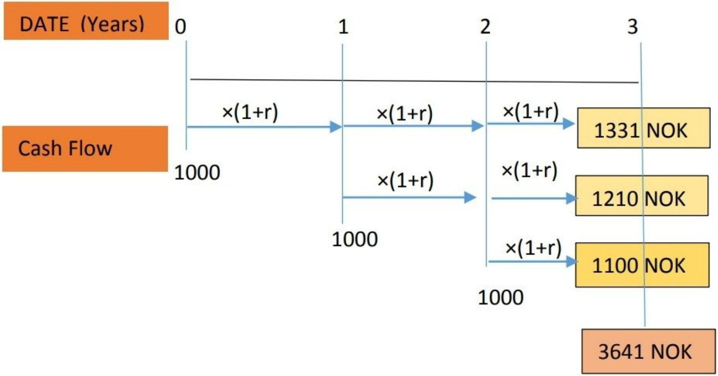
Cash Flow

Using the first two rules of time travel, we find that our total savings on date 1 will be 2100 NOK. Continuing in this fashion, we can solve the problem as follows:



Cash Flow

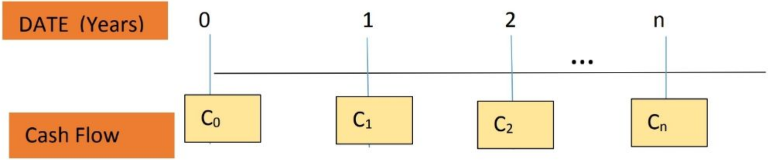
Another approach to the problem is to compute the future value in year 3 of each cash flow separately. We can then combine all three amounts.



Valuing a Stream of Cash Flows

Valuing a Stream of Cash Flows

Consider a stream of cash flows: C_0 at date 0, C_1 at date 1, and so on, up to C_n at date n. We represent this cash flow stream on a timeline as follows:



Valuing a Stream of Cash Flows

Using the time travel techniques, we compute the present value of this cash flow stream in two steps. First, we compute the present value of each individual cash flow. Then, once the cash flows are in common units of dollars today, we can combine them. For a given interest rate r, we represent this process as follows:

$$PV = C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + ... + \frac{C_n}{(1+r)^n}.$$

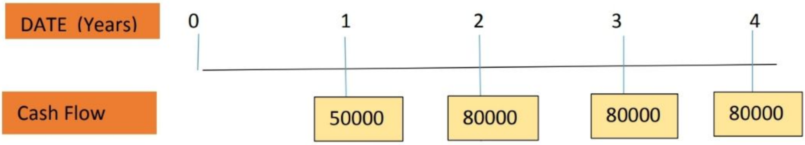
The present value is the amount you need to invest today to generate the cash flow stream C_0, C_1,..., C_n. That is, receiving those cash flows is equivalent to having their present value in the bank today.

Valuing a Stream of Cash Flows

Example

You have just graduated and need money to buy a new car. Your rich Uncle Henry will lend you the money so long as you agree to pay him back within 4 years, and you offer to pay him the rate of interest that he would otherwise get by putting his money in a savings account. Based on your earnings and living expenses, you think you will be able to pay him 50000 NOK in one year, and 80 000 NOK each year for the next three years. If Uncle Henry would otherwise earn 6 % per year on his savings, how much can you borrow from him?

Solution. The cash flows you can promise Uncle Henry are as follows:



Valuing a Stream of Cash Flows

- Uncle Henry should be willing to give you an amount that is equivalent to these payments in present value terms. This is the amount of money that it would take him to produce these same cash flows, which we calculate as follows:

$$V(0) = PV = \frac{50000}{(1+0.06)} + \frac{80000}{(1+0.06)^2} + \frac{80000}{(1+0.06)^3} + \frac{80000}{(1+0.06)^4} =$$

$$= 47169.8 + 71199.7 + 67169.5 + 63367.5 = 248906.5$$

  Thus, Uncle Henry should be willing to lend you 248906.5 in exchange for your promised payments.

- Let' s verify our answer. If your uncle kept his 248906.5 NOK in the bank today earning 6 % interest, in four years he would have

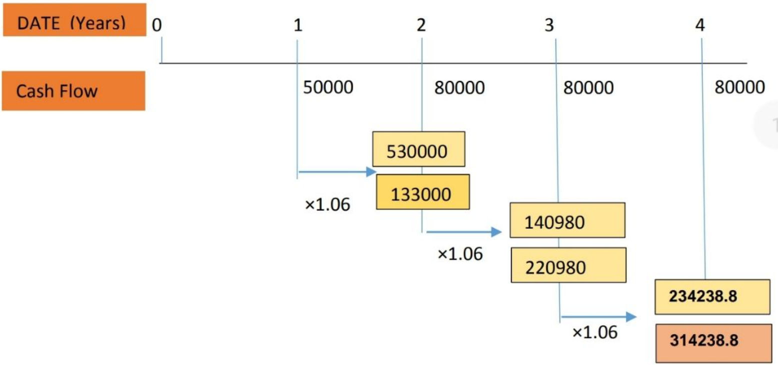
$$FV = 248906.5 \cdot (1+0.06)^4 = 314238.7$$

  in four years

Valuing a Stream of Cash Flows

Now suppose that Uncle Henry gives you the money, and then deposits your payments to him in the bank each year. *How much will he have four years from now?*

We need to compute the future value of the annual deposits. One way to do so is to compute the bank balance each year:



Valuing a Stream of Cash Flows

Rules

- The present value of a cash flow stream is

$$PV = \sum_{k=0}^{n} \frac{C_k}{(1+r)^k}.$$

- The present value equals the amount you would need in the bank today to recreate the cash flow stream.
- The future value on date n of a cash flow stream with a present value of PV is

$$FV = PV \cdot (1+r)^n.$$

Annuity

Annuity

Definition

Annuity —multiple payments of the same amount that occur over equal time periods.

Annuity

Examples of annuities include:

- the annual premium payable on a life insurance policy;
- interest payments on a bond and quarterly stock dividends.

The *payment interval* (or *period*) of an annuity is the time between successive payments, while the term is the time from the beginning of the first payment interval to the end of the last payment interval.

- An annuity is termed an *ordinary annuity* when payments are made at the same time that interest is credited, that is, at the *end* of the payment intervals.
- By contrast, an annuity with a periodic payment that is made at the beginning of each payment interval is known as an *annuity due*.
- If the payments begin and end on a fixed date, the annuity is known as *an annuity certain*.
- On the other hand, if the payments continue for ever, the annuity is known as *perpetuity*.

Annuity

- The amount or future value of an annuity is the sum of all payments made and the accumulated interest at the end of the term.
- The present value is the sum of payments, each discounted to the beginning of the term, that is, the sum of the present value of all payments.

Annuity

Ordinary annuities certain

Assume that each payment is C. Furthermore, suppose that the interest rate is r per period and that the number of periods in the term is n. The first payment of C will be compounded over $n-1$ periods.
and will accumulate to the value of V_1 at the end of the term where

$$V_1 = (1+r)^{n-1}.$$

The second payment will accumulate to V_2, where

$$V_2 = (1+r)^{n-2}.$$

And so on

$$V_3 = (1+r)^{n-3}, ..., V_{n-1} = (1+r), V_n = 1.$$

Annuity

The total amount of the annuity at the end of the term (after n payment intervals) is thus

$$V_1 + V_2 + ... + V_{n-1} + V_n =$$

$$= (1+r)^{n-1} + (1+r)^{n-2} + ... + (1+r) + 1.$$

This formula is also known as the future value of an annuity and is denoted by

$$\overline{s_n|}_r = \frac{(1+r)^n - 1}{r}.$$

Definition

If the payment in rand, made at each payment interval in respect of an ordinary annuity certain at interest rate r per payment interval, is C, then the amount or future value of the annuity after n intervals is

$$FV_{oac} = C \cdot \frac{(1+r)^n - 1}{r} \quad \text{or} \quad FV_{oac} = C \cdot \overline{s_n|}_r.$$

Annuity

Definition

The present value of an annuity is the amount of money that must be invested now, at r percent, so that n equal periodic payments may be withdrawn without any money being left over at the end of the term of n periods.

Definition

If the payment in rand made at each payment interval for an ordinary annuity certain, at interest rate r per payment interval, is C for a total of n payments, then the present value is

$$PV_{oac} = C \cdot \frac{(1+r)^n - 1}{r(1+r)^n},$$

where $\overline{a_n|}_r = \frac{(1+r)^n-1}{r(1+r)^n}$ is the present value of an annuity.

Annuity

Annuities due

Definition

An annuity-due is an annuity for which the payments are made at the beginning of the payment periods.

- The first payment is made at time 0, and the last payment is made at time $n-1$.
- We denote the present value of the annuity-due at time 0 by $\ddot{a}_n|_r$, and the future value of the annuity at time n by $\ddot{s}_n|_r$, where

$$\ddot{a}_n|_r = \frac{1-(1+r)^{-n}}{r}$$

and

$$\ddot{s}_n|_r = \frac{(1+r)^n - 1}{r \cdot (1+r)^{-1}}.$$

Annuity

As each payment in an annuity-due is paid one period ahead of the corresponding payment of an ordinary annuity, the present value of each payment in an annuity-due is $(1+r)$ times of the present value of the corresponding payment in an ordinary annuity. Hence,

$$\ddot{a}_{\overline{n}|r} = (1+r)a_{\overline{n}|r},$$

and, similarly,

$$\ddot{s}_{\overline{n}|r} = (1+r)s_{\overline{n}|r}.$$

Annuity

As an annuity-due of n payments consists of a payment at time 0 and an ordinary annuity of $n-1$ payments, the first payment of which is to be made at time 1, we have

$$\ddot{a}_{\overline{n}|r} = 1 + a_{\overline{n-1}|r}$$

Similarly, if we consider an ordinary annuity with $n+1$ payments at time $1, 2, ..., n+1$ as an annuity-due of n payments starting at time 1 plus a final payment at time $n+1$, we can conclude

$$s_{\overline{n+1}|r} = \ddot{s}_{\overline{n}|r} + 1.$$

Annuity

Example

A company wants to provide a retirement plan for an employee who is aged 55 now. The plan will provide her with an ordinary annuity of 70000 NOK every year for 15 years upon her retirement at the age of 65. The company is funding this plan with an annuity-due of 10 years. If the rate of interest is 5%, what is the amount of installment the company should pay?

Solution.

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Solution. Firstly calculate the present value of the retirement annuity

$$PV = C \cdot \overline{a_{15}|}_{0.05} = 70000 \cdot \frac{(1+0.05)^{15}-1}{0.05 \cdot (1+0.05)^{15}} = 726576.$$

Annuity

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$$PV = C \cdot \overline{a_{15}}|_{0.05} = 70000 \cdot \frac{(1+0.05)^{15}-1}{0.05\cdot(1+0.05)^{15}} = 726576.$$

This amount should be equal to the future value of the company's installments C^*, which is $C^* \cdot \ddot{s}_{10}|_{0.05}$:

$$\ddot{s}_{10}|_{0.05} = \frac{(1+0.05)^{10}-1}{0.05\cdot(1+0.05)^{-1}} = 13.2068, \text{ so that } C^* = \frac{726576}{13.2068} = 5502.$$

Annuity

Example

You are the lucky winner of the 30 million NOK state lottery. You can take your prize money either as (a) 30 payments of 1 million NOK per year (starting today), or (b) 15 million NOK paid today. If the interest rate is 8%, which option should you take?

Solution.

Annuity

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Solution. Because the first payment is paid today, the last payment will occur in 29 years (for a total of 30 payments). We can compute the present value:

$$PV = C \cdot \ddot{a}_{30}|_{0.08} = 1000000 \cdot \frac{1-(1+0.08)^{-30}}{0.08} = 12.16$$

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Therefore, the present value of option (a) is only 12.16 million NOK, and so it is more valuable to take option (b) and receive 15 million NOK upfront—even though we receive only half the total cash amount. The difference, of course, is due to the time value of money.

Deferred annuities and perpetuities

Deferred annuities and perpetuities

Definition

A *deferred annuity* is an annuity where the first payment is made a number of payment intervals after the end of the first interest period.

Essentially, however, this entails no additional complications. All you have to do is to work from a base date that is one period before the first payment so that ordinary annuities can be used and then discount to the actual beginning.

Deferred annuities and perpetuities

Example

Jonathan intends to start a dry cleaning business and wishes to borrow money for this purpose. He feels that he will not be able to repay anything for the first three years but, thereafter, he is prepared to pay back 200 000 NOK per year for five years. The bank agrees to advance him money at 18% interest per annum. How much would they be willing to advance him now under these conditions?

Solution.

Deferred annuities and perpetuities

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Solution. Taking year three as base date, the payments may be viewed as an ordinary annuity, with payments of 200000 NOK as stipulated.

$$PV_1 = C \cdot \overline{a_n|}_r = C \cdot \frac{(1+r)^n - 1}{r(1+r)^n},$$

Deferred annuities and perpetuities

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Solution. Taking year three as base date, the payments may be viewed as an ordinary annuity, with payments of 200000 NOK as stipulated.

$$PV_1 = C \cdot \overline{a_n}|_r = C \cdot \frac{(1+r)^n - 1}{r(1+r)^n},$$

$$PV_1 = C \cdot \overline{a_5}|_{0.18} = 200000 \cdot \frac{(1+0.18)^4 - 1}{0.18(1+0.18)^5} = 65434,2.$$

The present value on this base date is therefore $625434,2$ NOK. However, this is the value three years from now.

Deferred annuities and perpetuities

To obtain the present value PV (today or now) we have to discount this amount back for three years by using the compound interest formula and the stipulated interest rate.

$$PV = PV_1 \left(1 + \frac{r}{m}\right)^{-tm} = 65434,2 \cdot (1 + 0.18)^{-3}.$$

Deferred annuities and perpetuities

Definition

A *perpetuity is an annuity* with payments that begin on a fixed date and continue forever. That is, it is an annuity that does not stop.

Examples of perpetuities are

- scholarships paid from an endowment;
- the interest payments from an amount of money invested permanently;
- the dividends on a share, provided, of course, that the company does not cease to exist.

Deferred annuities and perpetuities

We cannot, of course, refer to the accumulated sum of a perpetuity, since the term of the perpetuity does not end. We can, however, calculate the present value of a perpetuity – it is simply the sum of the present values of the individual payments:

$$P_n = \frac{C}{(1+r)^n},$$

where C is the payment, r is the interest rate and n the period. We can then write the present value of the perpetuity, PV, as

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots$$

or

$$PV = \frac{C}{r}.$$

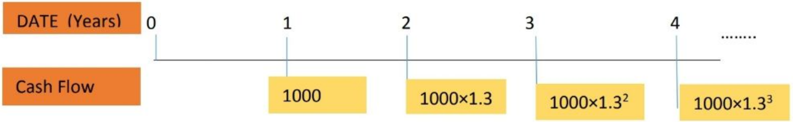
Growing Cash Flows

Deferred annuities and perpetuities

We have considered only cash flow streams that have the same cash flow every period. If, instead, the cash flows are expected to grow at a constant rate in each period, we can also derive a simple formula for the present value of the future stream.

Growing Perpetuity.

A growing perpetuity is a stream of cash flows that occur at regular intervals and grow at a constant rate forever. For example, a growing perpetuity with a first payment of 1000 NOK that grows at a rate of 3% has the following timeline:



Growing Cash Flows

In general, a growing perpetuity with a first payment C and a growth rate g will have the following series of cash flows:

As with perpetuities with equal cash flows, we adopt the convention that the first payment occurs at date 1. Note a second important convention: *The first payment does not include growth.*
That is, the first payment is C, even though it is one period away.
Similarly, the cash flow in period n undergoes only $n-1$ periods of growth.
Substituting the cash flows from the preceding timeline into the general formula for the present value of a cash flow stream gives

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \ldots, \text{ or } PV = \sum_{n=1}^{\infty} \frac{C(1+g)^{n-1}}{(1+r)^n}.$$

Growing Cash Flows

Suppose $g \geq r$. Then the cash flows grow even faster than they are discounted; each term in the sum gets larger, rather than smaller. In this case, the sum is infinite! What does an infinite present value mean? Remember that the present value is the ”do-it-yourself” cost of creating the cash flows. An infinite present value means that no matter how much money you start with, it is impossible to sustain a growth rate of g forever and reproduce those cash flows on your own. Growing perpetuities of this sort cannot exist in practice because no one would be willing to offer one at any finite price. A promise to pay an amount that forever grew faster than the interest rate is also unlikely to be kept (or believed by any savvy buyer). The only viable growing perpetuities are those where the perpetual growth rate is less than the interest rate, so that each successive term in the sum is less than the previous term and the overall sum is finite. Consequently, we assume that $g < r$ for a growing perpetuity.

Growing Cash Flows

Let's consider a specific case. Suppose you want to create a perpetuity with cash flows that grow by 2% per year, and you invest 1000 NOK in a bank account that pays 5% interest. At the end of one year, you will have 1050 NOK in the bank. If you withdraw only 30 NOK, you will have 1020 NOK to reinvest —2% more than the amount you had initially. This amount will then grow to

$$1020 \cdot 1.05 = 1071$$

in the following year, and you can withdraw

$$30 \cdot 1.02 = 30.6,$$

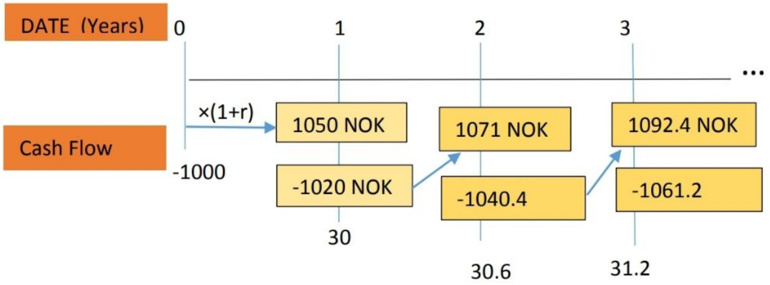
which will leave you with principal of 1071 NOK - 30.6 NOK $= 1040.4$. Note that

$$1020 \cdot 1.02 = 1040.4.$$

That is, both the amount you withdraw and the principal you reinvest grow by 2% each year.

Growing Cash Flows

On a timeline, these cash flows look like this:



By following this strategy, you have created a growing perpetuity that starts at 30 NOK and grows 2% per year. This growing perpetuity must have a present value equal to the cost of 1000 NOK.

Growing Cash Flows

We can generalize this argument. In the case of an equal-payment perpetuity, we deposited an amount P in the bank and withdrew the interest each year. Because we always left the principal P in the bank, we could maintain this pattern forever. If we want to increase the amount we withdraw from the bank each year by g, then the principal in the bank will have to grow by the same factor g. So, instead of withdrawing all of the interest rP, we leave gP in the bank in addition to our original principal P, and only withdraw

$$C = (r - g)P.$$

Solving this last equation for P, the initial amount deposited in the bank account, gives the present value of a growing perpetuity with initial cash flow C :

Present Value of a Growing Perpetuity

$$PV = \frac{C}{r - g}.$$

Thank you!