

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT4700 — Introduction to Mathematical Finance
and Investment Theory

Day of examination: Thursday January 18, 2024

Examination hours: 15.00 – 19.00

This problem set consists of 0 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All exercises count equally.

Problem 1

a

Let Ω be a finite probability space (sample space). Define what is an algebra on Ω and what is a filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}$.

b

Let Ω be a finite probability space (sample space) with a filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}$. Furthermore, let the asset price be given by an $\mathbb{F}$ -adapted stochastic process $(S(t))_{t=0}^T$ on Ω . with a filtration $\mathbb{F}$. What does it mean that $S(t)$ is $\mathcal{F}_t$ -measurable and that $(S(t))_{t=0}^T$ is $\mathbb{F}$ -adapted?

c

Consider a forward contract delivering the asset at time T . Argue that the forward price $F_t(T)$ of the contract at time $t \leq T$ must be

$$F_t(T) = S(t)(1+r)^{T-t}$$

where $r > 0$ is the interest rate.

d

Define what is a risk-neutral probability Q . If we have such Q , show that

$$F_t(T) = \mathbb{E}_Q[S(T) | \mathcal{F}_t].$$

(Continued on page 2.)

Problem 2

Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$ with a probability $P(\{\omega_i\}) = 1/6$ for $i = 1, \dots, 6$. Moreover, assume that $\mathcal{F} = \{\emptyset, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \Omega\}$.

a

Show that $\mathcal{F}$ is an algebra.

b

Let Z be a random variable such that $Z(\omega_i) = 1$ for $i = 1, 2, 6$ and zero otherwise. Show that Z is not $\mathcal{F}$ -measurable.

c

Compute $\mathbb{E}[Z|\mathcal{F}]$, and show that $\mathbb{E}[Z|\mathcal{F}]$ is $\mathcal{F}$ -measurable.

Problem 3

In this exercise you can argue by words and graphics.

a

Suppose you manage a portfolio of certain classes of stocks traded at an exchange. Explain, using the Markowitz theory, what happens with your portfolio's risk and expected return if you add another class of stocks traded at the exchange to your portfolio.

b

Suppose that you have access to bonds that give a return $r > 0$. Explain, using the Markowitz theory, how you can optimally construct a portfolio of stocks and bonds which achieves an expected return higher than r , but not more than r_T with $r_T > r$. Here r_T is the expected return of the tangent portfolio of the efficient frontier and σ_T is its risk.

Problem 4

In this exercise you are going to price options based on the Black & Scholes formula for at-the-money call options. We suppose that the continuously compounding interest rate is zero, $r = 0$ and we denote by S today's price of the underlying asset.

a

State the Black & Scholes formula for the price of a call option with exercise time T and strike $K = S$ (at-the-money call option). Denote this price by C .

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b

Denote the price of a put option with the same exercise time T and strike $K = S$ by P . Argue that buying the call and selling the put yields the same payoff as if you buy the stock and finance this by borrowing S in the bank. What is then the price of the put?

Problem 5

In a one-period market, suppose $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and that you have a price of an asset being $S(0) = 1$, $S(1)(\omega_1) = u$, $S(1)(\omega_2) = m$ and $S(1)(\omega_3) = d$. Here, $0 < d < 1 + r \leq m < u$, and the interest rate is given by $r > 0$.

a

Show that a claim paying X at time 1 is in general not possible to replicate.

b

Characterise all the risk-neutral probabilities Q on Ω .

Problem 6

Suppose the price of an asset $(S(t))_{t=0,1,2}$ is defined by a two-period binomial tree model, where the initial price $S(0) = 1$, and in each step the price goes up by a factor u or it remains the same. The interest rate is $r > 0$, and $u > 1 + r$. The probability space (sample space) is $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ in this model, where ω_1 corresponds to the asset price going up twice, ω_2 up and then down, ω_3 down and then up and finally ω_4 down twice.

a

Find $\mathcal{F}_1$, the algebra generated by $S(0)$ and $S(1)$. The risk-neutral probability Q is given by $Q(\omega_1) = q^2$, $Q(\omega_2) = q(1 - q) = Q(\omega_3)$, $Q(\omega_4) = (1 - q)^2$ and $q = \frac{r}{u-1}$. Suppose that you buy an option paying you 1 if the asset has gone down twice at time 2, and zero otherwise. Assume that the option is attainable. Under this assumption, find the value of the option at time 1 (without resorting to the replicating portfolio).

b

Find the replicating strategy for the option, and show that the value of this strategy at time 1 is the same as the one you derived in **b** above.

END

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