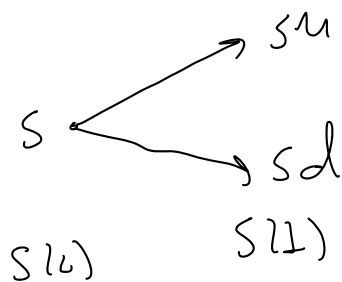


SUGGESTED SOLUTIONS EXAM AUTUMN 2023

STK-MAT 3700/4700

1



Interest rate is r .

$$\bar{X} = \begin{cases} x, & \text{if } s(1) = su \\ y, & \text{if } s(1) = sd \end{cases}$$

- a) Let a be the number of assets we buy or sell initially, and b how much money we borrow or deposit in the bank. At time 1 we require that

$$b(1+r) + a s(1) = \bar{X}$$

or, as equations

$$b(1+r) + a su = x$$

$$b(1+r) + a sd = y$$

A solution of this 2×2 system of equations, is

$$\underline{a = \frac{x-y}{s(u-d)}}, \quad \underline{b = \frac{yu-xd}{(1+r)(u-d)}}$$

The value of this portfolio today is

$$b + aS(t) = \frac{yu - xd}{(1+r)(u-d)} + \frac{x-y}{s(u-d)} \cdot s$$

$$= \frac{yu - xd + x(1+r) - y(1+r)}{(1+r)(u-d)}$$

$$= \frac{x(1+r-d) + y(u-(1+r))}{(1+r)(u-d)} \quad (= V_0)$$

Some may prefer to use the notation $H_0(1)$ for b and $H_1(1)$ for a .

b) We denote the price of the claim c , and $c > V_0$, where V_0 is today's value of the replicating portfolio.

Today:

Sell claim $\bar{X}$, $+ c$

Buy replicating portfolio $- V_0$

Place rest in bank $- (c - V_0)$

Net position

0

Time 1

Cover claim

$$\begin{cases} -x, S(1) = su \\ -y, S(1) = sd \end{cases}$$

$$\text{Sell replicating portfolio } \begin{cases} +x, S(1) = su \\ +y, S(1) = sd \end{cases}$$

Withdraw money
in bank

$$+ (c - V_0)(1+r)$$

Net position

$$+ (c - V_0)(1+r) > 0$$

This is a sure gain, hence an arbitrage opportunity.

c)

We have, by how q is defined,

$$\frac{1}{1+r} E_q[\bar{X}] = \frac{1}{1+r} (x - q + y(1-q))$$

$$\stackrel{!}{=} \frac{x(1+r-d) + y(u - (1+r))}{(1+r)(u-d)}$$

$$\text{or } (x-y)q = \frac{x(1+r-d) + y(u - (1+r))}{(u-d)} - y$$

$$\text{or } (x-y)q = \frac{(x-y)(1+r-d)}{(u-d)}$$

$$\text{or } \underline{\underline{q = \frac{1+r-d}{u-d}}} \quad (\text{This is known from the course!})$$

In the exercise, we have assumed $u > d$.

We check if q is a probability.

$$q > 0 \text{ iff } 1+r-d > 0 \Leftrightarrow d < 1+r$$

$$q < 1 \text{ iff } 1+r-d < u-d \Leftrightarrow u > 1+r.$$

We need to ADD the assumption that

$$\boxed{d < 1+r < u}$$

②

a) We do the buy&hold strategy to argue for the relationship:

$$\underline{F_0(T) > S(0)(1+r)^T}$$

Time 0:

Sell forward	0
Buy asset	$-S(0)$
Borrow in bank	$+S(0)$
	0

Time T :

Deliver asset, receive forward price $+F_0(T)$

Pay loan in bank $-S(0)(1+r)^T$

$$F_0(T) - S(0)(1+r)^T > 0$$

$$\underline{\underline{F_0(T) < S(0)(1+r)^T}}$$

Time 0

Buy forward 0

Short sell asset $+S(0)$

Deposit money in bank $-S(0)$

$$0$$

Time T :

Receive asset, which

closes the short position. Pay forward price $-F_0(T)$

$$\begin{array}{r} \text{Withdraw money} \\ \text{in bank} \end{array} + S(0)(1+r)^T$$

$$S(0)(1+r)^T - F_0(T) > 0$$

We have arbitrage opportunities whenever $F_0(T) \neq S(0)(1+r)^T$, but no arbitrage when $F_0(T) = S(0)(1+r)^T$

b)

Q is a risk-neutral probability if

(i) $Q(\omega) > 0$ for all $\omega \in \Omega$

(ii) $(S^*_t)_{t=0}^T$ is a Q -martingale, i.e.

$(S^*_t)_{t=0}^T$ is $\mathcal{F}$ -adapted and

$$E_Q[S^*_u | \mathcal{F}_t] = S^*_t, \quad u \geq t, \quad u, t = 0, \dots, T.$$

$$\text{where } S^*_t = S(t) / (1+r)^t,$$

If we have such a Q ;

$$E_Q[S^*(T)] = E_Q[S^*(T) | \mathcal{F}_0] = S^*(0) = S(0)$$

$$w, S(0) = \frac{1}{(1+r)^T} E_Q[S(T)]$$

$$\Rightarrow \underline{\underline{F_0(T) = E_Q[S(T)]}}$$

3

$$a) \quad \{\omega_1, \omega_2\}^c = \{\omega_3, \omega_4, \omega_5\} \in \mathcal{F}$$

$$\{\omega_3, \omega_4, \omega_5\}^c = \{\omega_1, \omega_2\} \in \mathcal{F}$$

$\Rightarrow \mathcal{F}$ is closed under complement

$$\{\omega_1, \omega_2\} \cup \{\omega_3, \omega_4, \omega_5\} = \Omega \in \mathcal{F}$$

$\Rightarrow \mathcal{F}$ is closed under union.

$\phi \in \mathcal{F}$. Hence, $\mathcal{F}$ is an algebra.

b) The partition of $\mathcal{F}$ is $A_1 = \{\omega_1, \omega_2\}$ and

$A_2 = \{\omega_3, \omega_4, \omega_5\}$. This is a partition because

$$A_1, A_2 \in \mathcal{F}, A_1 \cap A_2 = \phi, A_1 \cup A_2 = \Omega.$$

$$E[\Phi | \mathcal{F}](\omega) = \begin{cases} E[\Phi | A_1], & \omega \in A_1 \\ E[\Phi | A_2], & \omega \in A_2 \end{cases}$$

$$E[\Phi | A_1] = \sum_{i=1}^5 \Phi(\omega_i) P(\omega_i | A_1)$$

$$= \sum_{i=1}^5 \underbrace{\Phi(\omega_i)}_{=0, i=1,2,3} \frac{P(\{\omega_i\} \cap A_1)}{P(A_1)}$$

Bayes' Formula

$$= \underline{\underline{0}}, \text{ since } \{\omega_i\} \cap A_1 = \begin{cases} \{\omega_i\}, & i=1,2 \\ \emptyset, & i=3,4,5 \end{cases}$$

Likewise

$$E[\Phi | A_2] = \sum_{i=1}^5 \underbrace{\Phi(\omega_i)}_{=0, i=1,2,3} \frac{P(\{\omega_i\} \cap A_2)}{P(A_2)}$$

$$= 1 \cdot \frac{P(\{\omega_4\} \cap A_2)}{P(A_2)} + 1 \cdot \frac{P(\{\omega_5\} \cap A_2)}{P(A_2)}$$

$$= \frac{P(\omega_4)}{P(A_2)} + \frac{P(\omega_5)}{P(A_2)}$$

$$P(A_2) = P(\{\omega_3, \omega_4, \omega_5\}) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$$

$$= \frac{\frac{1}{5}}{\frac{3}{5}} + \frac{\frac{1}{5}}{\frac{3}{5}} = \underline{\underline{\frac{2}{3}}}$$

Hence

$$E[\mathbb{P}|\mathcal{F}](\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ 2/3, & \omega = \omega_3, \omega_4, \omega_5 \end{cases}$$

c)

$$\{\omega \in \Omega \mid E[\mathbb{P}|\mathcal{F}](\omega) = 0\} = \{\omega_1, \omega_2\} \in \mathcal{F}$$

$$\{\omega \in \Omega \mid E[\mathbb{P}|\mathcal{F}](\omega) = 2/3\} = \{\omega_3, \omega_4, \omega_5\} \in \mathcal{F}$$

Thus, $E[\mathbb{P}|\mathcal{F}]$ is constant on sets in $\mathcal{F}$, and therefore $\mathcal{F}$ -measurable.

4

$$a) \quad C(t) = S(t) \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

where $\Phi(x) = P(X \leq x)$, $X \sim N(0, 1)$, the cumulative standard normal distribution function,

$$\text{and} \quad d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln\left(\frac{S(t)}{K}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

b) From $C(u) - P(u) = S(u) - Ke^{-rT}$ we get

$$\begin{aligned} P(u) &= C(u) - S(u) + Ke^{-rT} \\ \underline{\underline{P(u)}} &= S(u) (\Phi(d_1) - 1) - Ke^{-rT} (\Phi(d_2) - 1) \end{aligned}$$

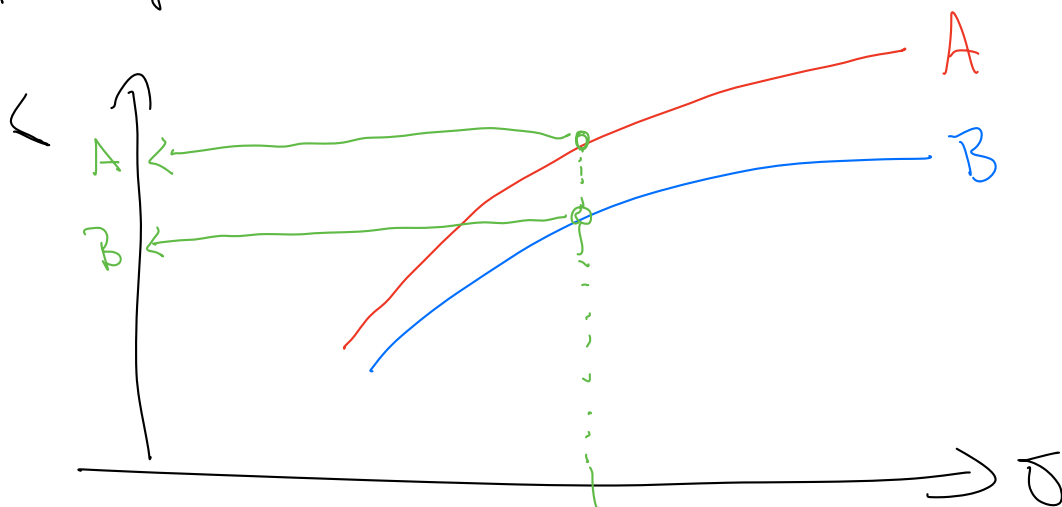
$$1 - \Phi(d) = 1 - P(X \leq d) = P(X \geq d)$$

$$= P(X \leq -d) = \Phi(-d)$$

$$= Ke^{-rT} \Phi(-d_2) - S(u) \Phi(-d_1)$$

⑤

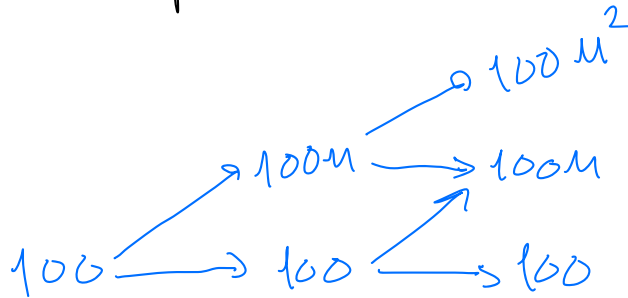
The more assets, the lower risk for a given expected return according to the Markowitz theory.



Investor A will get the highest return.

It is expected that students elaborate and discuss more!

(6)



ω_1 : up-up

ω_2 : up-down

ω_3 : down-up

ω_4 : down-down.

a) $S(t)$ generates only Ω , since

$$\{\omega \in \Omega \mid S(t)(\omega) = 100\} = \Omega.$$

$$\Rightarrow \mathcal{F}_0 = \{\emptyset, \Omega\}.$$

$$\{\omega \in \Omega \mid S(1)(\omega) = 100u\} = \{\omega_1, \omega_2\}$$

$$\{\omega \in \Omega \mid S(1)(\omega) = 100\} = \{\omega_3, \omega_4\}$$

The set $\mathcal{F}_1 = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$ is an

algebra:

$$(a) \emptyset \in \mathcal{F}_1$$

$$(b) \{\omega_1, \omega_2\}^c = \{\omega_3, \omega_4\} \in \mathcal{F}_1$$

$$\{\omega_3, \omega_4\}^c = \{\omega_1, \omega_2\} \in \mathcal{F}_1$$

$$(c) \{\omega_1, \omega_2\} \cup \{\omega_3, \omega_4\} = \Omega \in \mathcal{F}_1.$$

$$\Rightarrow \underline{\underline{\mathcal{F}_1 = \{ \emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega \}}}$$

3) $\bar{X}$ is the payoff of the digital:

$$\bar{X}(\omega) = \begin{cases} 100, & \omega = \omega_1 \\ 0, & \omega = \omega_2, \omega_3, \omega_4 \end{cases}$$

According to the pricing theorem for attainable claims,

$$C^*(1) = E_Q[\bar{X}^* | \mathcal{F}_1]$$

where $C(1)$ is the price of the digital at time 1, and

$C^*(1)$, $\bar{X}^*$ are discounted values.

i.e.,

$$\frac{C(1)}{(1+r)} = E_Q\left[\frac{\bar{X}}{(1+r)} \mid \mathcal{F}_1\right]$$

$$\Leftrightarrow C(1) = \frac{1}{1+r} E_Q[\bar{X} | \mathcal{F}_1].$$

We must calculate $E_Q[\bar{X} | \mathcal{F}_1]$: Partition of

$\mathcal{F}_1$ is $A_1 = \{\omega_1, \omega_2\}$, $A_2 = \{\omega_3, \omega_4\}$, since $A_1, A_2 \in \mathcal{F}_1$,
 $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = \Omega$.

$$E_q[X|\mathcal{F}_1](\omega) = \begin{cases} E_q[X|A_1], & \omega \in A_1 \\ E_q[X|A_2], & \omega \in A_2 \end{cases}$$

$X(\omega_1) = 100$, while $X(\omega_i) = 0$, $i = 3, 4$.

Hence;

$$E_q[X|A_k] = 100 \frac{Q(\{\omega_1\} \cap A_k)}{Q(A_k)}, \quad k = 1, 2.$$

But $\{\omega_1\} \cap A_2 = \emptyset$, so

$$E_q[X|A_2] = 0 \quad \text{and}$$

$$\begin{aligned} E_q[X|A_1] &= 100 \frac{Q(\{\omega_1\})}{Q(\omega_1) + Q(\omega_2)} = 100 \frac{q^2}{q^2 + q(1-q)} \\ &= 100 q = \frac{100r}{u-1}. \end{aligned}$$

$$\Rightarrow C(1) = \begin{cases} 100 \frac{r}{(1+r)(u-1)}, & \omega = \omega_1, \omega_2 \\ 0, & \omega = \omega_3, \omega_4. \end{cases}$$

c) We start with $t=1$, and find an investment in stock^(a) and bond^(b) replicating the digital at time $t=2$.

$$\underline{S(1) = 100u} \quad (w = w_1, w_2)$$

$$a \cdot 100u^2 + b(1+r) = 100$$

$$a \cdot 100u + b(1+r) = 0$$

$$\Rightarrow b = - \frac{a \cdot 100u}{1+r}$$

$$\Rightarrow a \cdot 100u^2 - a \cdot 100u = \cancel{100u} \cdot a(u-1) = \cancel{100}$$

$$\underline{a = \frac{1}{u(u-1)}}, \quad \underline{b = - \frac{100}{(1+r)(u-1)}}$$

$$\underline{S(1) = 100} \quad (w = w_3, w_4)$$

$$a \cdot 100u + b(1+r) = 0$$

$$a \cdot 100 + b(1+r) = 0$$

$$\Rightarrow \underline{a = b = 0}.$$

At time $t=1$, select

$$a = \begin{cases} \frac{1}{u(u-1)}, & w = w_1, w_2 \\ 0, & w = w_3, w_4 \end{cases}$$

$$b = \begin{cases} - \frac{100}{(1+r)(u-1)}, & w = w_1, w_2 \\ 0, & w = w_3, w_4 \end{cases}$$

We find $V(1)$:

$$V(1) = a \cdot S(1) + b$$

Inserting, we find

$$V_1(\omega) = \begin{cases} -\frac{100}{(1+r)(n-1)} + \frac{1}{n(n-1)} \cdot 100n & , \omega = \omega_1, \omega_2 \\ 0 & , \omega = \omega_3, \omega_4 \end{cases}$$

$$= \begin{cases} 100 \frac{r}{(1+r)(n-1)} & , \omega = \omega_1, \omega_2 \\ 0 & , \omega = \omega_3, \omega_4 \end{cases}$$

Note: $C(1) = V_1$!

We next go to time $t=0$, and find a (euros) and b (cents) that replicates $V(1)$!

$$\text{i.e., } a \cdot 100n + b(1+r) = \frac{100r}{(1+r)(n-1)}$$

$$a \cdot 100 + b(1+r) = 0$$

$$\Rightarrow \underline{a = \frac{r}{(1+r)(u-1)^2}} \quad , \quad \underline{b = -\frac{100r}{(1+r)^2(u-1)^2}}$$

We have found the replicating strategy.