

# Multiperiod Financial Markets

STK-MAT 3700/4700 An Introduction to Mathematical Finance

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## 1 Risk Neutral Pricing

# Risk Neutral Pricing

# Risk neutral pricing

A **contingent claim** is a random variable  $X$  representing the payoff at time  $T$  of a financial contract which depends on the values of the risky assets in the market.

## Example

Consider the market with  $T = 2$ ,  $K = 4$ ,  $S(0) = 5$ ,

$$S(1, \omega) = \begin{cases} 8 & \text{if } \omega = \omega_1, \omega_2 \\ 4 & \text{if } \omega = \omega_3, \omega_4 \end{cases}, S(2, \omega) = \begin{cases} 9 & \text{if } \omega = \omega_1 \\ 6 & \text{if } \omega = \omega_2, \omega_3 \\ 3 & \text{if } \omega = \omega_4 \end{cases}.$$

- $X = (S(2) - 5)^+$ . **European call option** with strike 5.

$$X = (\max(0, 9 - 5), \max(0, 6 - 5), \max(0, 6 - 5), \max(0, 3 - 5))^T = (4, 1, 1, 0)^T.$$

# Risk neutral pricing

## Example 1

- $Y = \left( \frac{1}{3} \sum_{t=0}^2 S(t) - 5 \right)^+$ . **Asian call option** with strike 5.

$$Y_1 = \left( \frac{1}{3} \sum_{t=0}^2 S(t, \omega_1) - 5 \right)^+ = \max \left( 0, \frac{1}{3} (5 + 8 + 9) - 5 \right) = 7/3,$$

$$Y_2 = \left( \frac{1}{3} \sum_{t=0}^2 S(t, \omega_2) - 5 \right)^+ = \max \left( 0, \frac{1}{3} (5 + 8 + 6) - 5 \right) = 4/3,$$

$$Y_3 = \left( \frac{1}{3} \sum_{t=0}^2 S(t, \omega_3) - 5 \right)^+ = \max \left( 0, \frac{1}{3} (5 + 4 + 6) - 5 \right) = 0,$$

$$Y_4 = \left( \frac{1}{3} \sum_{t=0}^2 S(t, \omega_4) - 5 \right)^+ = \max \left( 0, \frac{1}{3} (5 + 4 + 3) - 5 \right) = 0,$$

which yields  $Y = (7/3, 4/3, 0, 0)^T$ .

# Risk neutral pricing

**Standing Assumption:** The financial market model is arbitrage free. A contingent claim  $X$  is **attainable** (or **marketable**) if there exists a self-financing trading strategy such that  $V(T) = X$ .

Such strategy is said to replicate or generate or hedge  $X$ . The time  $t$  value of an attainable contingent claim  $X$ , denoted by  $P_X(t)$ , is equal to  $V(t)$ , the time  $t$  value of a portfolio generating  $X$ . Moreover,

$$V(t) = \mathbb{E}_Q \left[ \frac{B(t)}{B(T)} X \middle| \mathcal{F}_t \right], \quad t = 0, \dots, T, \quad Q \in \mathbb{M}.$$

# Risk neutral pricing

- In order to sell a contingent claim  $X$  the seller must find the trading strategy that replicates/hedges  $X$ .
- We will see three methods for finding a hedging strategy.

## First method

- We must know the value process  $V = \{V(t)\}_{t=0, \dots, T}$ .
- We solve

$$V(t) = H_0(t) + \sum_{n=1}^N H_n(t) S_n(t), \quad t = 1, \dots, T,$$

taking into account that  $H$  must be predictable.

# Risk neutral pricing

## Second method

- All we know is  $X$ .
- In this method, we work backwards in time and find  $V(t)$  and  $H(t)$  simultaneously.
- Since  $V(T) = X$ , we first find  $H(T)$  by taking into account that  $H$  is predictable and solving

$$X = H_0(T) B(T) + \sum_{n=1}^N H_n(T) S_n(T).$$

- Using that  $H$  must be self-financing, we find  $V(T-1)$  by computing

$$V(T-1) = H_0(T) B(T-1) + \sum_{n=1}^N H_n(T) S_n(T-1).$$

# Risk neutral pricing

## Second method

- Next, taking into account that  $H$  is predictable, we find  $H(T-1)$  by solving

$$V(T-1) = H_0(T-1)B(T-1) + \sum_{n=1}^N H_n(T-1)S_n(T-1).$$

- We repeat this procedure until computing  $V(0)$ .

# Risk neutral pricing

## Third method

- It relies on the fact that the self-financing condition

$$V^*(0) + G^*(t) = V^*(t),$$

is equivalent to

$$V^*(t-1) + \sum_{n=1}^N H_n(t) \Delta S_n^*(t) = V^*(t).$$

- We can use this system of equations, together with the predictability condition on  $H(t) = (H_1(t), \dots, H_N(t))^T$ , to find  $V^*(t-1)$  and  $H(t)$ .

# Risk neutral pricing

## Third method

- Then, we can find

$$H_0(t) = V^*(t-1) - \sum_{n=1}^N H_n(t) S_n^*(t-1),$$

$$V(t-1) = B(t-1) V^*(t-1).$$

- We begin with  $V^*(T) = X/B(T)$  and work backwards in time.

# Risk neutral pricing

## Example (Continuation Example 1)

Consider the market with  $T = 2$ ,  $K = 4$ ,  $S(0) = 5$ ,

$$S(1, \omega) = \begin{cases} 8 & \text{if } \omega = \omega_1, \omega_2 \\ 4 & \text{if } \omega = \omega_3, \omega_4 \end{cases}, \quad S(2, \omega) = \begin{cases} 9 & \text{if } \omega = \omega_1 \\ 6 & \text{if } \omega = \omega_2, \omega_3 \\ 3 & \text{if } \omega = \omega_4 \end{cases}.$$

- Suppose  $r = 0$ . We know that  $Q = (1/6, 1/12, 1/4, 1/2)^T$  is the unique martingale measure in this market.
- Consider  $X = (S(2) - 5)^+$  and  $Y = \left(\frac{1}{3} \sum_{t=0}^2 S(t) - 5\right)^+$  or in vector notation  $X = (4, 1, 1, 0)^T$  and  $Y = (7/3, 4/3, 0, 0)^T$ .
- Compute the price of  $X$  for each  $t$  and a self-financing trading strategy generating  $X$ . (Using the first and second methods)
- Do the same for  $Y$  using the third method.

# Thank you!