

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: 20 January 2023

Examination hours: 09.00–13.00

This problem set consists of 13 subsections. 4.2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

Solution Compute the expected return of the whole portfolio. The total portfolio value is

$$V = V_1 + V_2 + V_3 = 10000 \text{ NOK.}$$

The portfolio weights are

$$w_1 = \frac{V_1}{V} = \frac{1000}{10000} = 0.10, \quad w_2 = \frac{V_2}{V} = \frac{3500}{10000} = 0.35, \quad w_3 = \frac{V_3}{V} = \frac{5500}{10000} = 0.55.$$

The expected portfolio return is defined as the weighted average of the expected returns:

$$\mathbb{E}[R_p] = w_1\mathbb{E}[R_1] + w_2\mathbb{E}[R_2] + w_3\mathbb{E}[R_3].$$

Substituting the values,

$$\mathbb{E}[R_p] = 0.10 \cdot 0.10 + 0.35 \cdot 0.15 + 0.55 \cdot 0.05.$$

Compute each term:

$$0.10 \cdot 0.10 = 0.01, \quad 0.35 \cdot 0.15 = 0.0525, \quad 0.55 \cdot 0.05 = 0.0275.$$

Thus,

$$\mathbb{E}[R_p] = 0.01 + 0.0525 + 0.0275 = 0.09.$$

$$\boxed{\mathbb{E}[R_p] = 9\%}.$$

(Continued on page 2.)

b (weight 10p)

Solution

The variance of a two-asset portfolio is given by

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \rho_{12} \sigma_1 \sigma_2.$$

First note that

$$\sigma_1 = \sqrt{0.0004} = 0.02, \quad \sigma_2 = \sqrt{0.0009} = 0.03.$$

(a) Portfolio variance

We compute

$$\begin{aligned} \sigma_p^2 &= 0.4^2 \cdot 0.0004 + 0.6^2 \cdot 0.0009 + 2 \cdot 0.4 \cdot 0.6 \cdot 0.5 \cdot 0.02 \cdot 0.03 \\ &= 0.16 \cdot 0.0004 + 0.36 \cdot 0.0009 + 0.48 \cdot 0.5 \cdot 0.02 \cdot 0.03 \\ &= 0.000064 + 0.000324 + 0.000144 \\ &= 0.000532. \end{aligned}$$

Thus, the portfolio variance is

$$\sigma_p^2 = 0.000532.$$

c (weight 10p)

Strip: long 1 call + long 2 puts with strike K

The call and put payoffs are

$$C(S) = (S - K)^+ = \max(S - K, 0), \quad P(S) = (K - S)^+ = \max(K - S, 0).$$

The payoff of the strip is

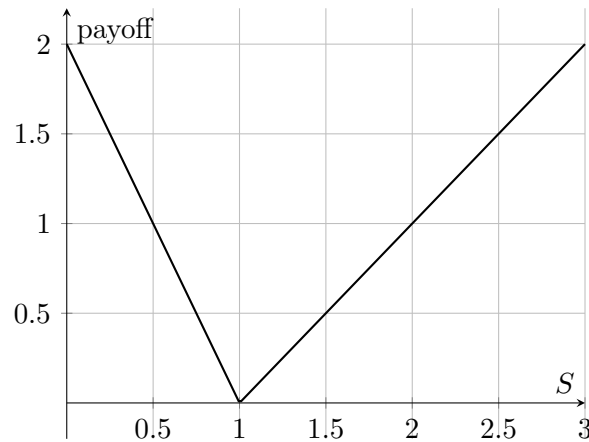
$$\pi_{\text{strip}}(S) = (S - K)^+ + 2(K - S)^+.$$

Equivalently, piecewise:

$$\pi_{\text{strip}}(S) = \begin{cases} 2(K - S), & S < K, \\ S - K, & S \geq K. \end{cases}$$

(Continued on page 3.)

Payoff diagram. Below is a payoff diagram for a strip with strike $K = 1$ (for illustration):



Problem 2

a (weight 10p)

Solution We assume $0 < d < 1 < u$, $S_0 = 1$, and the bond satisfies $B_t \equiv 1$.

Since $d^2 < K < ud$, the payoff of the call at time $t = 2$ is

$$C_2 = \begin{cases} u^2 - K, & S_2 = u^2, \\ ud - K, & S_2 = ud, \\ 0, & S_2 = d^2. \end{cases}$$

Step 1: Replication at time $t = 1$

At time 1 we consider a one-period portfolio consisting of

$$H_1^{(1)} \text{ (shares), } H_0^{(1)} \text{ (bond).}$$

We choose $(H_1^{(1)}, H_0^{(1)})$ separately in each node so that

$$H_1^{(1)} S_2 + H_0^{(1)} = C_2.$$

Node $S_1 = u$. Possible prices at $t = 2$ are u^2 and ud . We solve

$$\begin{cases} H_1^{(1,u)} u^2 + H_0^{(1,u)} = u^2 - K, \\ H_1^{(1,u)} ud + H_0^{(1,u)} = ud - K. \end{cases}$$

(Continued on page 4.)

Subtracting the second equation from the first gives

$$H_1^{(1,u)}(u^2 - ud) = u^2 - ud \Rightarrow H_1^{(1,u)} = 1.$$

Then

$$H_0^{(1,u)} = -K.$$

Hence the value of the replicating portfolio in this node is

$$V_1^{(u)} = H_1^{(1,u)}u + H_0^{(1,u)} = u - K.$$

Node $S_1 = d$. Possible prices at $t = 2$ are ud and d^2 . We solve

$$\begin{cases} H_1^{(1,d)}ud + H_0^{(1,d)} = ud - K, \\ H_1^{(1,d)}d^2 + H_0^{(1,d)} = 0. \end{cases}$$

Subtracting,

$$H_1^{(1,d)}(ud - d^2) = ud - K \Rightarrow H_1^{(1,d)} = \frac{ud - K}{d(u - d)}.$$

From the second equation,

$$H_0^{(1,d)} = -H_1^{(1,d)}d^2 = -\frac{d(ud - K)}{u - d}.$$

Thus the portfolio value in the down node is

$$\begin{aligned} V_1^{(d)} &= H_1^{(1,d)}d + H_0^{(1,d)} \\ &= \frac{ud - K}{d(u - d)}d - \frac{d(ud - K)}{u - d} \\ &= \frac{ud - K}{u - d} - \frac{d(ud - K)}{u - d} \\ &= \frac{(1 - d)(ud - K)}{u - d}. \end{aligned}$$

Step 2: Replication at time $t = 0$

At time 0 we hold

$$H_1^{(0)} \quad (\text{shares}), \quad H_0^{(0)} \quad (\text{bond}),$$

and require the value of this portfolio at time 1 to match the values $V_1^{(u)}$ and $V_1^{(d)}$ obtained above:

$$\begin{cases} H_1^{(0)}u + H_0^{(0)} = V_1^{(u)} = u - K, \\ H_1^{(0)}d + H_0^{(0)} = V_1^{(d)} = \frac{(1 - d)(ud - K)}{u - d}. \end{cases}$$

(Continued on page 5.)

Subtracting the second equation from the first:

$$H_1^{(0)}(u - d) = (u - K) - \frac{(1 - d)(ud - K)}{u - d},$$

so

$$H_1^{(0)} = \frac{u - K}{u - d} - \frac{(1 - d)(ud - K)}{(u - d)^2}.$$

Then

$$H_0^{(0)} = u - K - H_1^{(0)}u.$$

Step 3: Option price at time 0

Since $S_0 = 1$ and the bond is worth 1, the time-0 value of the replicating portfolio is

$$V_0 = H_1^{(0)}S_0 + H_0^{(0)} = H_1^{(0)} + H_0^{(0)}.$$

If the market price C_0 of the call differs from V_0 , an arbitrage is possible:

- If $C_0 > V_0$, short the option and buy the replicating portfolio, locking in $C_0 - V_0 > 0$.
- If $C_0 < V_0$, buy the option and short the replicating portfolio, locking in $V_0 - C_0 > 0$.

Therefore, by the absence of arbitrage,

$$\boxed{C_0 = V_0 = H_1^{(0)} + H_0^{(0)}}.$$

b (weight 10p)

Solution In the risk-neutral measure Q , the discounted stock price must be a martingale. Since $r = 0$, this means

$$S_0 = \mathbb{E}^Q[S_1].$$

Let q be the risk-neutral probability of an up-move in each period:

$$\mathbb{P}^Q(S_1 = u) = q, \quad \mathbb{P}^Q(S_1 = d) = 1 - q.$$

Then

$$1 = S_0 = \mathbb{E}^Q[S_1] = qu + (1 - q)d.$$

Solving for q gives

$$\boxed{q = \frac{1 - d}{u - d}}.$$

(Continued on page 6.)

Risk-neutral valuation

Under Q the probabilities of the terminal stock prices are

$$\mathbb{P}^Q(S_2 = u^2) = q^2, \quad \mathbb{P}^Q(S_2 = ud) = 2q(1 - q), \quad \mathbb{P}^Q(S_2 = d^2) = (1 - q)^2.$$

The payoff of the call is

$$C_2 = \begin{cases} u^2 - K, & S_2 = u^2, \\ ud - K, & S_2 = ud, \\ 0, & S_2 = d^2. \end{cases}$$

Therefore the risk-neutral expected payoff is

$$\mathbb{E}^Q[C_2] = q^2(u^2 - K) + 2q(1 - q)(ud - K).$$

Since $r = 0$, the present value is just this expectation, so the risk-neutral pricing formula gives

$$C_0 = \mathbb{E}^Q[C_2].$$

A straightforward algebraic simplification, using $q = \frac{1-d}{u-d}$, shows that this C_0 coincides with $V_0 = H_1^{(0)} + H_0^{(0)}$ computed in part (a). Hence the option price equals the present risk-neutral expectation of its payoff.

c (weight 10p)

Solution At time $t = 1$ we can also price the option under the risk-neutral measure Q .

Node $S_1 = u$

Conditional on $S_1 = u$, the stock at time 2 is either u^2 (up) or ud (down), with probabilities q and $1 - q$, respectively. Thus

$$\mathbb{E}^Q[S_2 \mid S_1 = u] = qu^2 + (1 - q)ud = u(qu + (1 - q)d) = u \cdot 1 = u,$$

since $qu + (1 - q)d = 1$ from part (b). Because in this node the payoff is always positive, we have

$$C_2 = S_2 - K \quad \text{when } S_1 = u,$$

so

$$C_1(u) = \mathbb{E}^Q[C_2 \mid S_1 = u] = \mathbb{E}^Q[S_2 \mid S_1 = u] - K = u - K.$$

But this is exactly the value of the replicating portfolio found in part (a):

$$C_1(u) = u - K = V_1^{(u)}.$$

(Continued on page 7.)

Node $S_1 = d$

Conditional on $S_1 = d$, the stock at time 2 is either ud (up) or d^2 (down). The payoff of the call is positive only in the up-state:

$$C_2 = \begin{cases} ud - K, & \text{with prob. } q, \\ 0, & \text{with prob. } 1 - q. \end{cases}$$

Therefore

$$C_1(d) = \mathbb{E}^Q[C_2 \mid S_1 = d] = q(ud - K) = \frac{1-d}{u-d}(ud - K) = \frac{(1-d)(ud - K)}{u - d}.$$

But this is exactly the value of the replicating portfolio in the down-node:

$$C_1(d) = V_1^{(d)}.$$

Conclusion

In each node at time 1 the option price obtained from the risk-neutral expectation coincides with the value of the replicating portfolio:

$$C_1(u) = V_1^{(u)}, \quad C_1(d) = V_1^{(d)}.$$

Hence the price of the option at time 1 equals the value of its replicating portfolio at time 1.

Problem 3

a (weight 10p)

The filtration generated by S_1 is

$$\mathcal{F}_0 = \sigma(S_1(0)) = \sigma(\{\Omega\}) = \{\emptyset, \Omega\},$$

$$\mathcal{F}_1 = \sigma(S_1(0), S_1(1)) = \sigma(\{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}) = \{\emptyset, \Omega, \{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\},$$

$$\mathcal{F}_2 = \sigma(S_1(0), S_1(1), S_1(2)) = \sigma(\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}) = \mathcal{P}(\Omega),$$

where $\mathcal{P}(\Omega)$ is the power set of Ω .

To compute $\mathbb{E}[S_1(2, \omega) \mid \mathcal{F}_1]$ we condition on the atoms of $\mathcal{F}_1$:

$$\mathbb{E}[S_1(2, \omega) \mid \mathcal{F}_1] = \mathbb{E}[S_1(2, \omega) \mid \{\omega_1, \omega_3\}] 1_{\{\omega_1, \omega_3\}} + \mathbb{E}[S_1(2, \omega) \mid \{\omega_2, \omega_4\}] 1_{\{\omega_2, \omega_4\}}.$$

Since $P(\omega_i) = \frac{1}{4}$ for all i ,

$$\mathbb{E}[S_1(2, \omega) \mid \{\omega_1, \omega_3\}] = \frac{S_1(2, \omega_1)P(\omega_1) + S_1(2, \omega_3)P(\omega_3)}{P(\omega_1) + P(\omega_3)} = \frac{7 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = 5,$$

(Continued on page 8.)

$$\mathbb{E}[S_1(2, \omega) \mid \{\omega_2, \omega_4\}] = \frac{S_1(2, \omega_2)P(\omega_2) + S_1(2, \omega_4)P(\omega_4)}{P(\omega_2) + P(\omega_4)} = \frac{3 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = 2.$$

Hence

$$\boxed{\mathbb{E}[S_1(2, \omega) \mid \mathcal{F}_1] = 5 \mathbf{1}_{\{\omega_1, \omega_3\}} + 2 \mathbf{1}_{\{\omega_2, \omega_4\}}}.$$

b (weight 20p)

Let $Q = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4})^T$ be the unique martingale measure on this market (you do not have to prove this). Consider the following optimal portfolio problem

$$\begin{aligned} \max_{H \in \mathbb{H}} \mathbb{E}[U(V(2))] \\ \text{subject to } V(0) = v, \end{aligned}$$

where v is a given strictly positive real number, $\mathbb{H}$ is the set of all self-financing and predictable trading strategies and $U(u) = 2u^{1/2}$. Compute the optimal attainable wealth, the optimal objective value and the optimal trading strategy.

(b) Optimal portfolio problem

Let $Q = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4})^T$ be the unique martingale measure. We consider

$$\max_{H \in \mathbb{H}} \mathbb{E}[U(V(2))] \quad \text{s.t.} \quad V(0) = v > 0,$$

with $U(u) = 2u^{1/2}$ and $\mathbb{H}$ the set of predictable self-financing strategies.

Since the market is arbitrage free and complete (unique martingale measure), we can use the martingale method.

Step 1: Optimal attainable wealth. Let L be the state price density

$$L = \frac{Q}{P} = \begin{pmatrix} \frac{1/8}{1/4} \\ \frac{1/4}{1/4} \\ \frac{3/8}{1/4} \\ \frac{1/4}{1/4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{3}{2} \\ 1 \end{pmatrix}, \quad B(2) = 1.$$

We solve the static problem

$$\max_W \mathbb{E}[U(W)] \quad \text{subject to} \quad \mathbb{E}^Q\left[\frac{W}{B(2)}\right] = v.$$

Here

$$U(u) = 2u^{1/2} \quad \Rightarrow \quad U'(u) = u^{-1/2}, \quad I(y) = (U')^{-1}(y) = y^{-2}.$$

(Continued on page 9.)

Hence the optimal attainable wealth is

$$\widehat{W} = I\left(\hat{\lambda} \frac{L}{B(2)}\right) = I(\hat{\lambda}L) = (\hat{\lambda}L)^{-2},$$

for some Lagrange multiplier $\hat{\lambda} > 0$.

Componentwise,

$$\widehat{W}(\omega_i) = (\hat{\lambda}L(\omega_i))^{-2} = \hat{\lambda}^{-2}L(\omega_i)^{-2}.$$

Since

$$L^{-2} = \begin{pmatrix} (1/2)^{-2} \\ 1^{-2} \\ (3/2)^{-2} \\ 1^{-2} \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ \frac{4}{9} \\ 1 \end{pmatrix},$$

we have

$$\widehat{W} = \hat{\lambda}^{-2} \begin{pmatrix} 4 \\ 1 \\ \frac{4}{9} \\ 1 \end{pmatrix}.$$

The constraint $\mathbb{E}^Q[\widehat{W}] = v$ gives

$$v = \hat{\lambda}^{-2} \sum_{i=1}^4 Q_i L(\omega_i)^{-2} = \hat{\lambda}^{-2} \left(\frac{1}{8} \cdot 4 + \frac{1}{4} \cdot 1 + \frac{3}{8} \cdot \frac{4}{9} + \frac{1}{4} \cdot 1 \right) = \hat{\lambda}^{-2} \cdot \frac{7}{6}.$$

Thus

$$\hat{\lambda}^{-2} = v \cdot \frac{6}{7},$$

and the optimal attainable wealth is

$$\widehat{W} = \frac{6v}{7} \begin{pmatrix} 4 \\ 1 \\ \frac{4}{9} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{24}{7}v \\ \frac{6}{7}v \\ \frac{8}{21}v \\ \frac{6}{7}v \end{pmatrix}.$$

Step 2: Optimal objective value. The optimal value of the objective function is

$$\begin{aligned} \mathbb{E}[U(\widehat{W})] &= \mathbb{E}[2\widehat{W}^{1/2}] = 2 \sum_{i=1}^4 P(\omega_i) \sqrt{\widehat{W}(\omega_i)} \\ &= 2 \cdot \frac{1}{4} \left(\sqrt{\frac{24}{7}v} + \sqrt{\frac{6}{7}v} + \sqrt{\frac{8}{21}v} + \sqrt{\frac{6}{7}v} \right). \end{aligned}$$

(Continued on page 10.)

Step 3: Optimal trading strategy $\widehat{H}$. We now find a predictable self-financing strategy

$$\widehat{H} = \{(H_0(t), H_1(t))^T\}_{t=1,2}$$

such that its value process V satisfies $V(2) = \widehat{W}$.

Since $r = 0$, the discounted and non-discounted prices coincide. The increments of the risky asset are

$$\Delta S_1(1) = S_1(1) - S_1(0) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \Delta S_1(2) = S_1(2) - S_1(1) = \begin{pmatrix} 3 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Time $t = 2$. Self-financing gives

$$\widehat{W} = V(2) = V(1) + H_1(2)\Delta S_1(2).$$

Because of predictability,

$$H_1(2, \omega_1) = H_1(2, \omega_3) =: h_u, \quad H_1(2, \omega_2) = H_1(2, \omega_4) =: h_d,$$

and

$$V(1, \omega_1) = V(1, \omega_3) =: x_u, \quad V(1, \omega_2) = V(1, \omega_4) =: x_d.$$

From $\widehat{W}$ we obtain the system

$$\begin{cases} \frac{24}{7}v = x_u + 3h_u & (\omega_1), \\ \frac{8}{21}v = x_u - h_u & (\omega_3), \end{cases} \quad \begin{cases} \frac{6}{7}v = x_d + h_d & (\omega_2), \\ \frac{6}{7}v = x_d - h_d & (\omega_4). \end{cases}$$

From the down-branch equations we get $h_d = 0$ and $x_d = \frac{6}{7}v$.

From the up-branch equations,

$$\left(\frac{24}{7} - \frac{8}{21}\right)v = 4h_u \quad \Rightarrow \quad h_u = \frac{16}{21}v,$$

and then $x_u = \frac{8}{7}v$.

Thus

$$V(1) = \begin{pmatrix} \frac{8}{7}v \\ \frac{6}{7}v \\ \frac{8}{7}v \\ \frac{6}{7}v \end{pmatrix}, \quad H_1(2) = \begin{pmatrix} \frac{16}{21}v \\ 0 \\ \frac{16}{21}v \\ 0 \end{pmatrix}.$$

Time $t = 1$. Predictability implies that $H_1(1)$ is constant (same for all ω). Self-financing gives

$$V(1) = V(0) + H_1(1)\Delta S_1(1),$$

(Continued on page 11.)

so for $\omega \in \{\omega_1, \omega_3\}$,

$$\frac{8}{7}v = v + H_1(1) \cdot 1,$$

and for $\omega \in \{\omega_2, \omega_4\}$,

$$\frac{6}{7}v = v + H_1(1) \cdot (-1).$$

Both equations yield

$$H_1(1) = \frac{1}{7}v, \quad V(0) = v.$$

Bond holdings. Finally, we obtain $H_0(1)$ and $H_0(2)$ from

$$V(t) = H_0(t)B(t) + H_1(t)S_1(t), \quad B(t) = 1.$$

At time 1:

$$H_0(1) = V(0) - H_1(1)S_1(0) = v - \frac{1}{7}v \cdot 3 = \frac{4}{7}v.$$

At time 2:

$$H_0(2, \omega) = V(1, \omega) - H_1(2, \omega)S_1(1, \omega) = \begin{cases} \frac{8}{7}v - \frac{16}{21}v \cdot 4 = -\frac{40}{21}v, & \omega \in \{\omega_1, \omega_3\}, \\ \frac{6}{7}v - 0 \cdot 2 = \frac{6}{7}v, & \omega \in \{\omega_2, \omega_4\}. \end{cases}$$

Thus the *optimal trading strategy* $\hat{H}$ is given by

$$\boxed{\begin{aligned} H_1(1) &= \frac{1}{7}v, & H_0(1) &= \frac{4}{7}v, \\ H_1(2, \omega) &= \begin{cases} \frac{16}{21}v, & \omega \in \{\omega_1, \omega_3\}, \\ 0, & \omega \in \{\omega_2, \omega_4\}, \end{cases} \\ H_0(2, \omega) &= \begin{cases} -\frac{40}{21}v, & \omega \in \{\omega_1, \omega_3\}, \\ \frac{6}{7}v, & \omega \in \{\omega_2, \omega_4\}. \end{cases} \end{aligned}}$$

and it replicates the optimal attainable wealth $\widehat{W}$ found above.

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space.

a (weight 10p)

Let $\{X_n, n \geq 1\}$ be independent identically distributed random variables such that $\mathbb{E}[X_i] = a, i \geq 1$. Set $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n), n \geq 1$. Find the conditional expectations:

$$\mathbb{E}[X_1 \cdot X_2 \cdot \dots \cdot X_n | \mathcal{F}_k], \quad n > k.$$

(Continued on page 12.)

b (weight 10p)

Solution.

To prove that G is a martingale we must show:

1. $G(t)$ is $\mathcal{F}_t$ -measurable for each t (i.e. G is $\mathbb{F}$ -adapted);
2. for all $t = 0, \dots, T-1$,

$$\mathbb{E}[G(t+1) \mid \mathcal{F}_t] = G(t) \quad (\text{martingale property}).$$

(We also tacitly assume integrability of all terms so that all expectations exist.)

1. *Adaptedness.*

First note that if X and Y are $\mathcal{G}$ -measurable, then $X + Y$ and XY are also $\mathcal{G}$ -measurable for any σ -algebra $\mathcal{G}$.

The process H is predictable, hence in particular adapted: $H(t)$ is $\mathcal{F}_{t-1}$ -measurable. The process Z is adapted because it is a martingale, so $Z(u)$ is $\mathcal{F}_u$ -measurable for every u . Moreover, $Z(u-1)$ is $\mathcal{F}_{u-1}$ -measurable and therefore also $\mathcal{F}_u$ -measurable, since $\mathbb{F}$ is a filtration and $\mathcal{F}_{u-1} \subseteq \mathcal{F}_u$.

Thus, for each fixed $u \geq 1$,

$$Z(u) - Z(u-1) \quad \text{and} \quad H(u)$$

are $\mathcal{F}_u$ -measurable, and hence their product

$$H(u)(Z(u) - Z(u-1))$$

is $\mathcal{F}_u$ -measurable.

For $t \geq 1$ we have

$$G(t) = \sum_{u=1}^t H(u)(Z(u) - Z(u-1)).$$

Each summand is $\mathcal{F}_u$ -measurable, and since $u \leq t$ we have $\mathcal{F}_u \subseteq \mathcal{F}_t$. Therefore each summand is $\mathcal{F}_t$ -measurable, and so is their finite sum $G(t)$. Also $G(0) = 0$ is trivially $\mathcal{F}_0$ -measurable. Hence G is $\mathbb{F}$ -adapted.

2. *Martingale property.*

For $t = 0, \dots, T-1$ we can write

$$G(t+1) = \sum_{u=1}^{t+1} H(u)(Z(u) - Z(u-1)) = G(t) + H(t+1)(Z(t+1) - Z(t)).$$

Taking conditional expectation with respect to $\mathcal{F}_t$ gives

$$\begin{aligned} \mathbb{E}[G(t+1) \mid \mathcal{F}_t] &= \mathbb{E}[G(t) + H(t+1)(Z(t+1) - Z(t)) \mid \mathcal{F}_t] \\ &= \mathbb{E}[G(t) \mid \mathcal{F}_t] + \mathbb{E}[H(t+1)(Z(t+1) - Z(t)) \mid \mathcal{F}_t] \\ &= G(t) + H(t+1) \mathbb{E}[Z(t+1) - Z(t) \mid \mathcal{F}_t], \end{aligned}$$

(Continued on page 13.)

where we used:

- linearity of conditional expectation in the second line;
- the fact that $G(t)$ is $\mathcal{F}_t$ -measurable in the third line;
- that $H(t+1)$ is $\mathcal{F}_t$ -measurable (predictability) so it can be taken out of the conditional expectation.

Since Z is a martingale,

$$\mathbb{E}[Z(t+1) \mid \mathcal{F}_t] = Z(t),$$

and hence

$$\mathbb{E}[Z(t+1) - Z(t) \mid \mathcal{F}_t] = 0.$$

Therefore,

$$\mathbb{E}[G(t+1) \mid \mathcal{F}_t] = G(t) + H(t+1) \cdot 0 = G(t).$$

Thus G satisfies the martingale property.

Since G is adapted and $\mathbb{E}[G(t+1) \mid \mathcal{F}_t] = G(t)$ for all $t = 0, \dots, T-1$, we conclude that

G is a martingale with respect to $\mathbb{F}$.
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