

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: 17 Nov 2025

Examination hours: 12.00–14.00

This problem set consists of 3 subsection.4.3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

Consider a portfolio consisting of three stock positions. Their market values are

$$V_1 = 1000 \text{ NOK}, \quad V_2 = 3500 \text{ NOK}, \quad V_3 = 5500 \text{ NOK}.$$

The expected returns of the three assets are

$$\mathbb{E}[R_1] = 10\%, \quad \mathbb{E}[R_2] = 15\%, \quad \mathbb{E}[R_3] = 5\%.$$

Compute the expected return of the whole portfolio.

b (weight 10p)

A portfolio consists of two risky assets with current market values $V_1 = 400 \text{ NOK}$, $V_2 = 600 \text{ NOK}$. The variances of the asset returns are

$$\sigma_1^2 = 0.0004, \quad \sigma_2^2 = 0.0009,$$

and the correlation coefficient between the two asset returns is $\rho_{12} = 0.5$. Compute the variance of the portfolio return.

(Continued on page 2.)

c (weight 10p)

Write the payoff functions (as functions of the stock price S) for the following securities:

1. A *strip*: long 1 call option and long 2 put options with the same strike K .
2. A *bull spread*: long 1 call option with strike K_1 and short 1 call option with strike K_2 , where $K_1 < K_2$.

For each case, draw the payoff diagram.

Problem 2

Suppose a stock price dynamics is given by a two-period binomial model. Today's stock price is $S(0) = 1$, and in each period it goes up by a factor u and down by d , with $0 < d < 1 < u$. The interest rate is $r = 0$. Assume there is a European call option written on the stock with strike K , paying $(S_2 - K)^+$ in period two. Suppose that the strike satisfies

$$d^2 < K < ud.$$

a (weight 15p)

Derive the replicating portfolio of the call option. Argue by arbitrage that the price of the option must be equal to the cost of the replicating portfolio.

b (weight 15p)

What is the risk-neutral probability q for a stock price increase in each period? Show that the present expected value of the option payoff is equal to the option price, where the expectation is taken with respect to the risk-neutral probability.

Problem 3

Consider a two-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, probability measure $P = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$, interest rate $r = 0$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1,2}$ with prices given by

$$S_1(0) = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} 4 \\ 2 \\ 4 \\ 2 \end{pmatrix}, \quad S_1(2) = \begin{pmatrix} 7 \\ 3 \\ 3 \\ 1 \end{pmatrix}.$$

a (weight 10p)

Find $\{\mathcal{F}_t\}_{t=0,1,2}$, the filtration generated by the price process $S_1 = \{S_1(t)\}_{t=0,1,2}$. Calculate $\mathbb{E}[S_1(2, \omega) | \mathcal{F}_1]$.

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b (weight 20p)

Let $Q = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4})^T$ be the unique martingale measure on this market (you do not have to prove this). Consider the following optimal portfolio problem

$$\begin{aligned} \max_{H \in \mathbb{H}} \mathbb{E}[U(V(2))] \\ \text{subject to } V(0) = v, \end{aligned}$$

where v is a given strictly positive real number, $\mathbb{H}$ is the set of all self-financing and predictable trading strategies and $U(u) = 2u^{1/2}$. Compute the optimal attainable wealth, the optimal objective value and the optimal trading strategy.

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space.

a (weight 10p)

Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion and let $a > 0$ be a constant. Define the process

$$X(t) := \frac{1}{a} B(a^2 t), \quad t \geq 0.$$

The time scale of the Brownian motion is compressed by a factor a^2 , while its magnitude is scaled by the factor $1/a$. Find the expected value and the variance of $X(t)$. Determine the probability distribution and the probability density function of $X(t)$. Finally, prove that the process $\{X(t) : t \geq 0\}$ is itself a standard Brownian motion.

b (weight 10p)

Let $\{X_n, n \geq 1\}$ be independent identically distributed random variables such that $\mathbb{E}[X_i] = a, i \geq 1$. Set $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n), n \geq 1$. Find the conditional expectations:

$$\mathbb{E}[X_1 \cdot X_2 \cdot \dots \cdot X_n | \mathcal{F}_k], \quad n > k.$$

c (weight 10p)

Define what is a martingale with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0, \dots, T}$ under the probability measure P . Let $Z = \{Z(t)\}_{t=0, \dots, T}$ be a martingale and $H = \{H(t)\}_{t=1, \dots, T}$ be a predictable process. Prove that the process $G = \{G(t)\}_{t=0, \dots, T}$ defined by

$$G(0) = 0,$$

$$G(t) = \sum_{u=1}^t H(u)(Z(u) - Z(u-1)), \quad t = 1, \dots, T.$$

is also a martingale.

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