

Notes on Risk–Neutral Computational Approach

Example

$$N = 2, \quad K = 3, \quad \Omega = \{w_1, w_2, w_3\}, \quad \pi = \frac{1}{9}, \quad B(1) = \frac{10}{9}, \quad S_1^*(0) = 6, \quad S_2^*(0) = 10.$$

$$S_1^*(1) = \begin{cases} 6, & w = w_1, \\ 8, & w = w_2, \\ 4, & w = w_3, \end{cases} \quad S_2^*(1) = \begin{cases} 13, & w = w_1, \\ 9, & w = w_2, \\ 8, & w = w_3. \end{cases}$$

with $P = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^\top$.

One can check that $Q = (1/3, 1/3, 1/3)^\top \in \mathcal{H}$.

Consider the utility function

$$U(u) = -e^{-u}, \quad U'(u) = e^{-u}.$$

At a maximum point

$$\begin{aligned} 0 &= \frac{\partial}{\partial H_1} \mathbb{E}[U(B(1)(V + H_1 \Delta S_1^* + H_2 \Delta S_2^*))] = \mathbb{E}[\Delta S_1^* \exp\{-B(1)(V + H_1 \Delta S_1^* + H_2 \Delta S_2^*)\}], \\ 0 &= \frac{\partial}{\partial H_2} \mathbb{E}[U(B(1)(V + H_1 \Delta S_1^* + H_2 \Delta S_2^*))] = \mathbb{E}[\Delta S_2^* \exp\{-B(1)(V + H_1 \Delta S_1^* + H_2 \Delta S_2^*)\}]. \end{aligned}$$

Thus, one has to solve two nonlinear equations for H_1 and H_2 ; in general this requires numerical methods.

Consider Risk–neutral computational approach

The risk–neutral computational approach consists of two steps:

Step 1. Maximize $\mathbb{E}[V(1)]$ over the set W_V of feasible random variables $V(1)$; i.e., determine the optimal terminal wealth $V(1)$ such that $V(0) = v$.

Step 2. Given the optimal terminal wealth $V(1)$, determine a trading strategy that generates it.

If $U(u) = e^{-u}$, then $I(y) = -\log y$.

$$\widehat{W} = -\log(\widehat{\lambda}LB^{-1}(1)) = -\log \widehat{\lambda} - \log(LB^{-1}(1)).$$

The multiplier satisfies

$$v = \mathbb{E}_Q \left[\frac{\widehat{W}}{B(1)} \right] = -\mathbb{E}_Q [B^{-1}(1) \log \widehat{\lambda}] - \mathbb{E}_Q [B^{-1}(1) \log(LB^{-1}(1))],$$

hence

$$\widehat{\lambda} = \exp \left\{ -v - \frac{\mathbb{E}_Q [B^{-1}(1) \log(LB^{-1}(1))]}{\mathbb{E}_Q [B^{-1}(1)]} \right\}.$$

Substituting back,

$$\begin{aligned} \widehat{W} &= -\log \exp \left\{ -v - \frac{\mathbb{E}_Q [B^{-1}(1) \log(LB^{-1}(1))]}{\mathbb{E}_Q [B^{-1}(1)]} \right\} - \log(LB^{-1}(1)) \\ &= v + \frac{\mathbb{E}_Q [B^{-1}(1) \log(LB^{-1}(1))]}{\mathbb{E}_Q [B^{-1}(1)]} - \log(LB^{-1}(1)). \end{aligned}$$

The corresponding utility value equals

$$U(\widehat{W}) = -\widehat{\lambda} L B^{-1}(1).$$

Specialization to Example

Therefore

$$\mathbb{E}[U(\widehat{W})] = -\widehat{\lambda} \mathbb{E}[L B^{-1}] = -\widehat{\lambda} \mathbb{E}_Q[B^{-1}].$$

If $B(0) = 1$ and $B(1) = 1 + r$, then

$$\mathbb{E}[U(\widehat{W})] = -\frac{\widehat{\lambda}}{1+r}.$$

Assume now the setup of Example 5.11 with

$$P = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)^\top, \quad Q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)^\top, \quad L = \left(\frac{2}{3}, \frac{4}{3}, \frac{4}{3} \right)^\top, \quad B(1) = \frac{10}{9}.$$

Compute

$$\mathbb{E}_Q \left[\log \left(\frac{L}{B(1)} \right) \right] = \frac{1}{3} \log \left(\frac{2/3}{10/9} \right) + \frac{1}{3} \log \left(\frac{4/3}{10/9} \right) + \frac{1}{3} \log \left(\frac{4/3}{10/9} \right) = \frac{1}{3} \log \left(\frac{108}{125} \right).$$

Hence

$$\widehat{\lambda} = \exp \left\{ -\frac{10}{9}v - \frac{1}{3} \log \left(\frac{108}{125} \right) \right\},$$

and the optimal expected utility is

$$\mathbb{E}[U(\widehat{W})] = \frac{9}{10} \exp \left\{ -\frac{10}{9}v - \frac{1}{3} \log \left(\frac{108}{125} \right) \right\}.$$

The optimal wealth components are

$$\begin{aligned} \widehat{W}_1 &= \frac{10}{9}v + \frac{1}{3} \log \left(\frac{108}{125} \right) - \log \left(\frac{3}{5} \right), \\ \widehat{W}_2 &= \frac{10}{9}v + \frac{1}{3} \log \left(\frac{108}{125} \right) - \log \left(\frac{6}{5} \right), \\ \widehat{W}_3 &= \frac{10}{9}v + \frac{1}{3} \log \left(\frac{108}{125} \right) - \log \left(\frac{6}{5} \right). \end{aligned}$$

Discounted optimal wealth:

$$\begin{aligned}\widehat{W}_1^* &= v + \frac{3}{10} \log\left(\frac{108}{125}\right) - \frac{9}{10} \log\left(\frac{3}{5}\right), \\ \widehat{W}_2^* &= v + \frac{3}{10} \log\left(\frac{108}{125}\right) - \frac{9}{10} \log\left(\frac{6}{5}\right), \\ \widehat{W}_3^* &= v + \frac{3}{10} \log\left(\frac{108}{125}\right) - \frac{9}{10} \log\left(\frac{6}{5}\right).\end{aligned}$$

In general

$$V(1) = v + G^* = v + H_1 \Delta S_1^* + H_2 \Delta S_2^*, \quad \Delta S_1^* = (0, 2, -2)^\top, \quad \Delta S_2^* = (3, -1, -2)^\top.$$

Solving for the strategy H

From $\widehat{W}^*$ we obtain the system

$$\begin{cases} \frac{3}{10} \log\left(\frac{108}{125}\right) - \frac{9}{10} \log\left(\frac{3}{5}\right) = -H_1 + 3H_2, \\ \frac{3}{10} \log\left(\frac{108}{125}\right) - \frac{9}{10} \log\left(\frac{6}{5}\right) = 12 - H_1 - 2H_2, \\ \frac{3}{10} \log\left(\frac{108}{125}\right) - \frac{9}{10} \log\left(\frac{6}{5}\right) = -2H_1 - 2H_2. \end{cases}$$

Hence

$$\begin{aligned}H_2 &= \frac{1}{10} \log\left(\frac{108}{125}\right) - \frac{3}{10} \log\left(\frac{3}{5}\right) \approx 0.143863, \\ H_1 &= \frac{1}{5} \log\left(\frac{108}{125}\right) - \frac{9}{20} \log\left(\frac{6}{5}\right) - \frac{3}{20} \log\left(\frac{3}{5}\right) \approx -0.034466.\end{aligned}$$

Finally, using $v = V(0) = H_0 + H_1 S_1(0) + H_2 S_2(0) = H_0 + 6H_1 + 10H_2$,

$$H_0 = v - 1.17834.$$