STK-MAT3700/4700
Introduction to Mathematical Finance
EXERCISE SET 10

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Schedule note: This set is presented in Week 45, Thursday 5 November 2026. It corresponds to Lecture 13.

Main topics: CRR time discretization, recombining trees, risk-neutral CRR probabilities, log-return decompositions, standardized increments, the central-limit intuition, and the real-world versus risk-neutral continuous-time stock-price formulas. Main book sections: Petters–Dong, Secs. 5.2.1–5.3.2.

Exercise 1. CRR Scaling and the Risk-Neutral Log Drift

Source basis: Based on Petters–Dong, Sec. 5.2.2, The Risk-Neutral CRR Tree, pp. 230–236, and Petters–Dong, Sec. 5.3.2, The Continuous-Time Security Price Formula, pp. 241–245.

Consider a CRR tree over a time interval of length $\tau = 1$ (time span) with

$$n = 16 \quad \text{(number of time steps)},$$

$$\sigma = 0.24 \quad \text{(volatility)}, \qquad r = 0.04 \quad \text{(risk-free rate)},$$

and

$$\delta = 0.01 \quad \text{(continuous dividend yield)}.$$

Use

$$h_n = \frac{\tau}{n}, \qquad u_n = e^{\sigma\sqrt{h_n}}, \qquad d_n = e^{-\sigma\sqrt{h_n}}.$$

a) Compute h_n, u_n, and d_n.

b) Check that the tree recombines, meaning that an up move followed by a down move gives the same stock price as a down move followed by an up move. Also check the no-arbitrage inequality

$$d_n < e^{(r-\delta)h_n} < u_n.$$

c) Compute the risk-neutral up-probability p_n^* and $1 - p_n^*$.

d) Compute the risk-neutral one-step log drift

$$\mu_n^* = \frac{p_n^* \log u_n + (1 - p_n^*) \log d_n}{h_n}.$$

e) Compare your answer with the limiting drift

$$\mu^* = r - \delta - \frac{\sigma^2}{2}.$$

Date: Autumn 2026.

Exercise 2. From Products to Log Sums

Source basis: Based on Petters–Dong, Sec. 5.2.1, The Real-World CRR Tree, especially equation (5.44), p. 230, and Lecture 13.

Let $S_0 = 100$ (initial stock price), $\tau = 1$ (time span), and $n = 4$ (number of time steps). Hence $h_n = 1/4$. Suppose

$$u_n = e^{0.1} \quad \text{(up factor)}, \qquad d_n = e^{-0.1} \quad \text{(down factor)},$$

and consider the path $UUDD$.

For the real-world probability, suppose the up-probability is

$$p_n = 0.55 \quad \text{(physical up-probability)}.$$

a) Compute the stock prices along the path $UUDD$.

b) Compute the one-step log returns

$$L_{n,j} = \log\left(\frac{S(t_j)}{S(t_{j-1})}\right), \qquad j = 1, 2, 3, 4.$$

c) Compute μ_n and σ_n from

$$\mathbb{E}[L_{n,j}] = \mu_n h_n, \qquad \mathrm{Var}(L_{n,j}) = \sigma_n^2 h_n.$$

d) Compute the standardized increments

$$X_{n,j} = \frac{L_{n,j} - \mu_n h_n}{\sigma_n \sqrt{h_n}}$$

along the path $UUDD$, and then compute

$$Z_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_{n,j}.$$

e) Verify the discrete CRR log-price formula

$$S(t_n) = S_0 e^{\mu_n \tau + \sigma_n \sqrt{\tau} Z_n}.$$

Exercise 3. Standardized Increments and the CLT Intuition

Source basis: Based on Petters–Dong, Exercise 5.21, p. 251, and Petters–Dong, Sec. 5.3.1, The Lindeberg Central Limit Theorem, pp. 237–242. This is a guided version adapted to the notation in Lecture 13.

Fix the number of time steps n in a CRR approximation. For each step $j = 1, \ldots, n$, the one-step log return $L_{n,j}$ takes two values, corresponding to an up move or a down move. Let

$$\mathbb{E}[L_{n,j}] = \mu_n h_n \quad (\mu_n \text{ is the log-return drift rate}),$$

and

$$\mathrm{sd}(L_{n,j}) = \sigma_n \sqrt{h_n} \quad (\sigma_n \text{ is the volatility scale}).$$

and define

$$X_{n,j} = \frac{L_{n,j} - \mu_n h_n}{\sigma_n \sqrt{h_n}}.$$

Assume the up-probability is p_n and the down-probability is $1 - p_n$.

a) Show that $X_{n,j}$ takes the two values

$$\frac{1-p_n}{\sqrt{p_n(1-p_n)}} \quad \text{and} \quad -\frac{p_n}{\sqrt{p_n(1-p_n)}}.$$

b) Verify that $\mathbb{E}[X_{n,j}] = 0$.
c) Verify that $\mathrm{Var}(X_{n,j}) = 1$.
d) If $p_n \to 1/2$, what do the two possible values of $X_{n,j}$ converge to?
e) Explain briefly why this standardization is useful when forming

$$Z_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_{n,j}.$$

Exercise 4. Real-World Versus Risk-Neutral Distributions

Source basis: Based on Petters–Dong, Sec. 5.3.2, The Continuous-Time Security Price Formula, pp. 241–245, especially equations (5.67)–(5.70). Related old exam questions use binomial trees and later Black–Scholes pricing; this question isolates the distributional step before the Black–Scholes formula.

Let

$$S_0 = 100 \quad \text{(initial stock price)},$$
$$m = 0.08 \quad \text{(real-world total return rate)},$$
$$r = 0.03 \quad \text{(risk-free rate)},$$
$$\delta = 0.01 \quad \text{(continuous dividend yield)},$$

and

$$\sigma = 0.20 \quad \text{(volatility)}.$$

Take $t = 1$ (time to maturity) and $K = 110$ (strike level).

a) Compute the real-world log-price drift

$$\mu_{RW} = m - \delta - \frac{\sigma^2}{2}$$

and the risk-neutral log-price drift

$$\mu^* = r - \delta - \frac{\sigma^2}{2}.$$

b) Write the real-world and risk-neutral continuous-time distribution formulas for $S(1)$.
c) Compute $\mathbb{E}_P[S(1)]$ and $\mathbb{E}_Q[S(1)]$.
d) Write $P(S(1) > K)$ under the real-world probability and under the risk-neutral probability in terms of the standard normal distribution function Φ. Then give numerical approximations.
e) Compute the normal-coordinate drift shift

$$Z_1^* = Z_1 + \frac{m-r}{\sigma}.$$