STK-MAT3700/4700
Introduction to Mathematical Finance
EXERCISE SET 11

KENNETH H. KARLSEN

Schedule note: This set is presented in Week 46, Thursday 12 November 2026. It corresponds mainly to Lecture 14, with exam-style use of the formulas from Lecture 15.

Main topics: BSM marketplace assumptions, Itô formula, self-financing replication, the BSM p.d.e., terminal and boundary conditions, the BSM call formula, put-call parity, and put and straddle formulas. Main book sections: Petters–Dong, Secs. 8.1.1–8.2.2.

Exercise 1. Model Ingredients and the BSM P.D.E.

Source basis: Based on Petters–Dong, Sec. 8.1.1, Marketplace Assumptions, pp. 385–386; Petters–Dong, Sec. 8.1.2, Money Market Account and the Underlier Model, pp. 386–388; and Petters–Dong, Sec. 8.1.4, Derivation of the BSM p.d.e., pp. 390–391.

In the BSM model, the money market account and the ex-dividend underlier price satisfy

$$dB_t = rB_t\,dt, \qquad dS_t = (m-\delta)S_t\,dt + \sigma S_t\,dW_t.$$

Here r is the risk-free rate, m is the real-world total-return rate, δ is the continuous dividend yield, and σ is the volatility.

Let $f(x,t)$ be the price at time t of a derivative written on the underlier, evaluated at underlier price x.

a) State the BSM p.d.e. satisfied by f.
b) Which parameter in the stock-price model does not appear in the BSM p.d.e.?
c) Explain briefly why this is important from a no-arbitrage pricing point of view.
d) Write the BSM p.d.e. in the special case $\delta = 0$.

Exercise 2. Deriving the BSM P.D.E.

Source basis: Based on Petters–Dong, Sec. 8.1.2, Money Market Account and the Underlier Model, pp. 386–388; Petters–Dong, Sec. 8.1.3, Self-Financing, Replicating Portfolio, pp. 388–390; and Petters–Dong, Sec. 8.1.4, Derivation of the BSM p.d.e., pp. 390–391.

Assume that

$$dS_t = (m-\delta)S_t\,dt + \sigma S_t\,dW_t$$

and that a smooth derivative price has the form $f(S_t,t)$. The underlier pays continuous dividends at rate δ.

a) Use Itô's formula to write $df(S_t,t)$.

Date: Autumn 2026.

b) A self-financing portfolio holding n_t units of the underlier and b_t units of the money market account has value

$$V_t = n_t S_t + b_t B_t.$$

Write the differential dV_t, including the dividend income.

c) Choose $n_t = f_x(S_t, t)$ and impose $V_t = f(S_t, t)$. Write $b_t B_t$ in terms of f, S_t, and f_x.

d) Compare the dt-terms and derive the BSM p.d.e.

Exercise 3. Conditions and the Parity Building Block

Source basis: Based on Petters–Dong, Sec. 8.1.4, pp. 390–391; Petters–Dong, Sec. 8.2.1, Solving the BSM p.d.e. for European Calls, pp. 392–395; and Petters–Dong, Sec. 8.2.2, BSM Pricing Formula for European Puts, pp. 395–396.

Let $\tau = T - t$. A European call and a European put have the same strike K and maturity T.

a) Write the terminal payoff of the call and the put.

b) Write the boundary condition at $x = 0$ for the call and the put.

c) Let

$$G(x, t) = xe^{-\delta\tau} - Ke^{-r\tau}.$$

Verify that G satisfies the BSM p.d.e.

d) Compute $G(x, T)$ and explain how this is related to put-call parity.

e) For the no-dividend case $\delta = 0$, make a qualitative plot of the terminal call payoff $x \mapsto (x - K)^+$ and the time-t call price $x \mapsto C^E(x, t)$ in the same coordinate system.

Exercise 4. At-the-Money Call Without a Normal Table

Source basis: Based on Petters–Dong, Sec. 8.2.1, equations (8.33)–(8.38), pp. 394–395. Inspired by the 2024 exam question asking for a compact Black–Scholes call formula in the at-the-money case.

Assume there are no dividends, $\delta = 0$, and the interest rate is $r = 0$. Let the current stock price and strike be equal:

$$S_0 = K = S.$$

The maturity is T and the volatility is σ.

a) State the BSM formula for a European call in the no-dividend case.

b) Compute d_+ and d_- when $S_0 = K$ and $r = 0$.

c) Show that the call price can be written as

$$C_0^E = S\left[2N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right].$$

Exercise 5. Put-Call Parity, Put Formula, and a Straddle

Source basis: Based on Petters–Dong, Sec. 8.2.2, BSM Pricing Formula for European Puts, pp. 395–396. Inspired by the 2023 exam question on deriving the put formula from put-call parity, and by the 2023 trial exam question on straddles.

Consider a European call and put with the same underlier, strike K, maturity T, risk-free rate r, dividend yield δ, and volatility σ. Let $x = S_t$ and $\tau = T - t$.

a) State put-call parity.
b) Starting from the BSM call formula, derive the BSM put formula.
c) A straddle is a long position in one call and one put with the same K and T. Express the straddle price $C^E + P^E$ in terms of C^E, x, K, r, δ, and τ.
d) Express the straddle price directly using $N(d_+)$ and $N(d_-)$.