

STK-MAT3700/4700
Introduction to Mathematical Finance
FORMULAS: THE BINOMIAL MODEL

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11. THE BINOMIAL MODEL

Source: Pliska, Sec. 3.5, The Binomial Model, pp. 100–105; Lecture 11 (Week 42, Tuesday 13 October 2026).

Path Space and Information

Binomial path space.

$$\Omega = \{0, 1\}^T, \quad \omega = (\omega_1, \dots, \omega_T)$$

The value $\omega_t = 1$ means an up move, and $\omega_t = 0$ means a down move.

Bernoulli move process.

$$X_t(\omega) = \omega_t, \quad P(X_t = 1) = p, \quad P(X_t = 0) = 1 - p$$

Number of up moves.

$$N_t = X_1 + \dots + X_t$$

Time- t information set.

$$F_t(\omega) = \{\eta \in \Omega : \eta_1 = \omega_1, \dots, \eta_t = \omega_t\}$$

Binomial filtration.

$$\mathcal{P}_t = \{F_t(\omega) : \omega \in \Omega\}, \quad \mathcal{F}_t = \sigma(\mathcal{P}_t)$$

Binomial Probabilities and Stock Prices

Binomial distribution under P .

$$P(N_t = n) = \binom{t}{n} p^n (1-p)^{t-n}, \quad n = 0, \dots, t$$

Mean and variance of up moves.

$$\mathbb{E}[N_t] = tp, \quad \text{Var}(N_t) = tp(1-p)$$

Binomial stock price.

$$S_t = S_0 u^{N_t} d^{t-N_t}, \quad t = 0, \dots, T$$

Equivalently, if $N_t = n$, then

$$S_t = S_0 u^n d^{t-n}.$$

Distribution of the stock price under P .

$$P(S_t = S_0 u^n d^{t-n}) = \binom{t}{n} p^n (1-p)^{t-n}$$

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Return Process

One-period stock return.

$$\Delta R_1(t) = \frac{S_t - S_{t-1}}{S_{t-1}}, \quad 1 + \Delta R_1(t) = \frac{S_t}{S_{t-1}}$$

Binomial return process.

$$\Delta R_1(t) = u^{X_t} d^{1-X_t} - 1$$

Thus

$$\Delta R_1(t) = \begin{cases} u - 1, & X_t = 1, \\ d - 1, & X_t = 0. \end{cases}$$

Risk-Neutral Probability and No-Arbitrage

Binomial risk-neutral probability.

$$q = \frac{1 + r - d}{u - d}, \quad 1 - q = \frac{u - 1 - r}{u - d}$$

Binomial no-arbitrage condition.

$$d < 1 + r < u$$

One-step martingale condition.

$$\mathbb{E}_Q[S_{t+1} \mid \mathcal{F}_t] = (1 + r)S_t$$

Equivalently, in discounted form,

$$\mathbb{E}_Q[S_{t+1}^* \mid \mathcal{F}_t] = S_t^*.$$

Martingale Measure on Paths

Path probability under Q .

$$Q(\omega) = q^n (1 - q)^{T-n}$$

Here ω has n up moves and $T - n$ down moves.

Binomial distribution under Q .

$$Q(N_t = n) = \binom{t}{n} q^n (1 - q)^{t-n}$$

Stock-price distribution under Q .

$$Q(S_t = S_0 u^n d^{t-n}) = \binom{t}{n} q^n (1 - q)^{t-n}$$

One-Step Replication

At a node with current stock price $S_t = s$, a one-step claim has up value X^u and down value X^d . The value formula is

$$V_{t+1} = h_0 B_{t+1} + h_1 S_{t+1}.$$

Replication requires

$$h_0 B_{t+1} + h_1 u s = X^u, \quad h_0 B_{t+1} + h_1 d s = X^d.$$

One-step replicating holdings.

$$h_1 = \frac{X^u - X^d}{s(u - d)}, \quad h_0 = \frac{uX^d - dX^u}{(u - d)B_{t+1}}$$

Backward Pricing**Terminal condition.**

$$V_T = X$$

Risk-neutral value process.

$$V_t = \mathbb{E}_Q \left[\frac{X}{(1 + r)^{T-t}} \mid \mathcal{F}_t \right], \quad t = 0, \dots, T$$

Backward pricing recursion.

$$V_t = \frac{qV_{t+1}^u + (1 - q)V_{t+1}^d}{1 + r}, \quad t = T - 1, \dots, 0$$