STK-MAT3700/4700
Introduction to Mathematical Finance
FORMULAS: TWO-PERIOD BINOMIAL PRICING AND CRR TREES

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12. Two-Period Binomial Pricing and CRR Trees

Source: Petters–Dong, Sec. 8.5.3, pp. 417–420; Petters–Dong, Sec. 5.2.2, pp. 230–236; Pliska, Sec. 3.5, pp. 100–105; Lecture 12 (Week 43, Tuesday 20 October 2026).

Pliska and Petters–Dong Notation

In Lecture 11, the bank account and discounted variables were

$$B_t = (1+r)^t, \qquad S_t^* = \frac{S_t}{B_t}, \qquad V_t^* = \frac{V_t}{B_t}.$$

Discounted-stock martingale.

$$\mathbb{E}_Q[S_{t+1}^* \mid \mathcal{F}_t] = S_t^*$$

Without dividends, this is equivalent to

$$\mathbb{E}_Q[S_{t+1} \mid \mathcal{F}_t] = (1+r)S_t.$$

Discounted value process.

$$V_t^* = \mathbb{E}_Q\left[\frac{X}{B_T} \mid \mathcal{F}_t\right]$$

Equivalently,

$$V_t = \frac{1}{(1+r)^{T-t}}\mathbb{E}_Q[X \mid \mathcal{F}_t].$$

Petters–Dong use continuously compounded time steps. If $t_j - t_0 = jh$, then

$$B(t_j) = e^{rjh}, \qquad S^*(t_j) = e^{-rjh}S(t_j).$$

They write risk-neutral expectation as $\mathbb{E}^*[\cdot]$.

Petters–Dong one-step stock condition.

$$\mathbb{E}^*[S(t_{j+1}) \mid \mathcal{F}_j] = e^{(r-\delta)h}S(t_j)$$

Here δ is the continuous dividend yield. Cash payoffs are still discounted at the risk-free rate r:

$$C(t_j) = e^{-rh}\mathbb{E}^*[C(t_{j+1}) \mid \mathcal{F}_j].$$

One-Step Risk-Neutral Probability

In one step, the stock moves from S_0 to S_0u or S_0d. Let p^* be the risk-neutral probability of an up move.

Risk-neutral probability with dividend yield.

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

Date: Autumn 2026.

Equivalently, p^* is determined by

$$p^*u + (1-p^*)d = e^{(r-\delta)h}.$$

Petters–Dong no-arbitrage condition.

$$d < e^{(r-\delta)h} < u$$

Two-Period Tree and Filtration

For a two-period recombining tree, the terminal paths are

$$\Omega = \{UU, UD, DU, DD\}.$$

Here U and D are path labels, while u and d are stock-price multipliers.

Two-period information nodes.

$$F_{1,1} = A_U = \{UU, UD\}, \qquad F_{1,2} = A_D = \{DU, DD\}$$

The corresponding time-1 sigma-algebra is

$$\mathcal{F}_1 = \{\emptyset, F_{1,1}, F_{1,2}, \Omega\}.$$

European Call Payoffs

Let K be the strike and let the option expire at t_2.

Terminal two-period call payoffs.

$$C^E_{u^2}(t_2) = \max\{S_0u^2 - K, 0\}, \qquad C^E_{ud}(t_2) = \max\{S_0ud - K, 0\}$$

The third terminal payoff is

$$C^E_{d^2}(t_2) = \max\{S_0d^2 - K, 0\}.$$

Direct Two-Period Pricing

Direct two-period European call price.

$$C^E(t_0;2) = e^{-2rh}\Big[(p^*)^2C^E_{u^2}(t_2) + 2p^*(1-p^*)C^E_{ud}(t_2) + (1-p^*)^2C^E_{d^2}(t_2)\Big]$$

The coefficient $2p^*(1-p^*)$ appears because the two paths UD and DU lead to the same middle terminal stock price S_0ud.

Backward Induction

Upper time-t_1 node value.

$$C^E_u(t_1) = e^{-rh}\left[p^*C^E_{u^2}(t_2) + (1-p^*)C^E_{ud}(t_2)\right]$$

Lower time-t_1 node value.

$$C^E_d(t_1) = e^{-rh}\left[p^*C^E_{ud}(t_2) + (1-p^*)C^E_{d^2}(t_2)\right]$$

Time-t_0 value by backward induction.

$$C^E(t_0;2) = e^{-rh}\left[p^*C^E_u(t_1) + (1-p^*)C^E_d(t_1)\right]$$

Risk-Neutral CRR Tree

In the Cox–Ross–Rubinstein tree, h_n is the time-step length and σ is the stock volatility. The one-step factors are chosen approximately symmetrically in log scale.

CRR up and down factors.

$$u_n \approx e^{\sigma\sqrt{h_n}}, \qquad d_n \approx e^{-\sigma\sqrt{h_n}}$$

Thus

$$u_n d_n \approx 1.$$

CRR risk-neutral probability.

$$p_n^* = \frac{e^{(r-\delta)h_n} - d_n}{u_n - d_n}$$

CRR no-arbitrage condition.

$$d_n < e^{(r-\delta)h_n} < u_n$$

Terminal stock price after k up moves.

$$S_T(k) = S_0 u_n^k d_n^{n-k}$$

CRR European payoff pricing sum.

$$V_0 = e^{-rT} \sum_{k=0}^{n} \binom{n}{k} (p_n^*)^k (1-p_n^*)^{n-k} f(S_0 u_n^k d_n^{n-k})$$