STK-MAT3700/4700
Introduction to Mathematical Finance
FORMULAS: CONTINUOUS-TIME LIMIT OF CRR SECURITY PRICES

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13. Continuous-Time Limit of CRR Security Prices

Source: Petters–Dong, Sec. 5.3.2, The Continuous-Time Security Price Formula, pp. 241–245; supporting material from Petters–Dong, Sec. 5.2.1, The Real-World CRR Tree, pp. 219–230; Petters–Dong, Sec. 5.2.2, The Risk-Neutral CRR Tree, pp. 230–236; Petters–Dong, Sec. 5.3.1, The Lindeberg Central Limit Theorem, pp. 237–242; Lecture 13 (Week 44, Tuesday 27 October 2026).

CRR Inputs

CRR time step.

$$h_n = \frac{\tau}{n}$$

CRR up and down factors.

$$u_n \approx e^{\sigma\sqrt{h_n}}, \qquad d_n \approx e^{-\sigma\sqrt{h_n}}$$

Real-world and risk-neutral drifts.

$$\mu_{RW} = m - \delta - \frac{\sigma^2}{2}, \qquad \mu^* = r - \delta - \frac{\sigma^2}{2}$$

Risk-neutral one-step log drift.

$$p_n^* = \frac{e^{(r-\delta)h_n} - d_n}{u_n - d_n}, \qquad \mu_n^* = \frac{p_n^* \log u_n + (1 - p_n^*) \log d_n}{h_n}$$

Risk-neutral drift limit.

$$\mu_n^* = r - \delta - \frac{\sigma^2}{2} + O(h_n), \qquad \mu_n^* \to \mu^*$$

One-Step Log Returns

One-step log return.

$$L_{n,j} = \log\left(\frac{S(t_j)}{S(t_{j-1})}\right)$$

Mean and variance of one-step log return.

$$\mathbb{E}[L_{n,j}] = \mu_n h_n, \qquad \mathrm{Var}(L_{n,j}) = \sigma_n^2 h_n$$

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Standardized one-step log return.

$$X_{n,j} = \frac{L_{n,j} - \mu_n h_n}{\sigma_n \sqrt{h_n}}$$

Normalized sum.

$$Z_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_{n,j}$$

Discrete CRR Security Price

Discrete CRR price formula.

$$S^{(n)}(t_n^{(n)}) = S_0 e^{\mu_n \tau + \sigma_n \sqrt{\tau} Z_n}$$

Central limit theorem input.

$$Z_n \xrightarrow{d} Z, \qquad Z \sim N(0,1)$$

Continuous-Time Security Prices

Real-world terminal convergence.

$$S^{(n)}(t_n^{(n)}) \xrightarrow{d} S(t_f),$$
$$S(t_f) \stackrel{d}{=} S_0 e^{\mu_{RW} \tau + \sigma \sqrt{\tau} Z}$$

Real-world continuous-time formula.

$$S(t) \stackrel{d}{=} S_0 e^{\mu_{RW} t + \sigma \sqrt{t}\, Z_t}, \qquad Z_t \sim N(0,1)$$

Risk-neutral continuous-time formula.

$$S(t) \stackrel{d}{=} S_0 e^{\mu^* t + \sigma \sqrt{t}\, Z_t^*}, \qquad Z_t^* \sim N(0,1)$$

Expected real-world price.

$$\mathbb{E}[S(t)] = S_0 e^{(m-\delta)t}$$

Expected risk-neutral price.

$$\mathbb{E}^*[S(t)] = S_0 e^{(r-\delta)t}$$

Change from Real-World to Risk-Neutral

Normal-coordinate shift.

$$Z_t^* = Z_t + \frac{\mu_{RW} - \mu^*}{\sigma} \sqrt{t} = Z_t + \frac{m - r}{\sigma} \sqrt{t}$$

Under its own probability description, Z_t is standard normal and Z_t^* is also standard normal. The formula above is a coordinate relation between the real-world and risk-neutral descriptions of the same stock-price level.

Geometric Brownian Motion Notation

Real-world GBM form.

$$S(t) = S(0) e^{\mu_{RW} t + \sigma B(t)}, \qquad B(t) \stackrel{d}{=} \sqrt{t}\, Z_t$$

Risk-neutral GBM form.

$$S(t) = S(0) e^{\mu^* t + \sigma B^*(t)}, \qquad B^*(t) \stackrel{d}{=} \sqrt{t}\, Z_t^*$$