STK-MAT3700/4700
Introduction to Mathematical Finance
FORMULAS: THE BSM MODEL AND THE BSM PDE

KENNETH H. KARLSEN

14. The BSM Model and the BSM PDE

Source: Petters–Dong, Sec. 8.1, The BSM Model, pp. 384–391; Lecture 14 (Week 45, Tuesday 3 November 2026).

Petters–Dong writes Brownian motion as a script-style $\mathcal{B}_t$ and writes the dividend yield as q. These notes use W_t for Brownian motion and δ for dividend yield, while B_t denotes the money market account.

Money Market Account

Money market account.

$$B_t = B_0 e^{rt}$$

Money market account dynamics.

$$dB_t = rB_t dt$$

Underlier Model

Real-world lognormal underlier.

$$S_t = S_0 e^{\mu_{RW} t + \sigma W_t}, \qquad \mu_{RW} = m - \delta - \frac{\sigma^2}{2}$$

Ex-dividend underlier dynamics.

$$dS_t = (m - \delta) S_t dt + \sigma S_t dW_t$$

Cum-dividend underlier.

$$S_t^c = e^{\delta t} S_t$$

Cum-dividend underlier dynamics.

$$dS_t^c = m S_t^c dt + \sigma S_t^c dW_t$$

Derivative Price

Derivative price as a deterministic function.

$$f_t = f(S_t, t)$$

Ito Formula

Deterministic chain rule.

$$\frac{d}{dt} g(X_t) = g'(X_t) \frac{dX_t}{dt}, \qquad dg(X_t) = g'(X_t) dX_t$$

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Stochastic correction for $g(X_t)$.

$$dg(X_t) = g'(X_t)dX_t + \frac{1}{2}g''(X_t)(dX_t)^2$$

Formal differential rules.

$$(dt)^2 = 0, \qquad dt\,dW_t = 0, \qquad (dW_t)^2 = dt$$

Squared increment for an SDE.

$$dX_t = u_t dt + v_t dW_t,$$
$$(dX_t)^2 = v_t^2 dt$$

Time-independent Ito formula.

$$dg(X_t) = \left[u_t g'(X_t) + \frac{1}{2}v_t^2 g''(X_t)\right] dt + v_t g'(X_t) dW_t$$

Time-dependent correction.

$$dg(X_t, t) = g_x(X_t, t)dX_t + g_t(X_t, t)dt + \frac{1}{2}g_{xx}(X_t, t)(dX_t)^2$$

Time-dependent Ito formula.

$$dg(X_t, t) = \left[\frac{1}{2}v^2(X_t, t)g_{xx}(X_t, t) + u(X_t, t)g_x(X_t, t) + g_t(X_t, t)\right] dt + v(X_t, t)g_x(X_t, t)dW_t$$

Ito formula for the derivative price.

$$df(S_t, t) = \left[\frac{1}{2}\sigma^2 S_t^2 f_{xx}(S_t, t) + (m - \delta)S_t f_x(S_t, t) + f_t(S_t, t)\right] dt + \sigma S_t f_x(S_t, t)dW_t$$

Self-Financing Replication

Replicating portfolio value.

$$V_t = n_t S_t^c + b_t B_t = f(S_t, t)$$

Self-financing condition.

$$dV_t = n_t dS_t^c + b_t dB_t$$

Bank-account holding from replication.

$$b_t = \frac{f(S_t, t) - n_t S_t^c}{B_t}$$

Portfolio differential from self-financing.

$$dV_t = [rf(S_t, t) + n_t(m - r)S_t^c]\,dt + n_t \sigma S_t^c dW_t$$

Delta and Holdings

Risky holding from matching dW_t terms.

$$n_t = \frac{S_t}{S_t^c} f_x(S_t, t)$$

Delta.

$$\Delta_f(S_t, t) = f_x(S_t, t)$$

Bank-account holding in delta form.

$$b_t = \frac{f(S_t, t) - S_t \Delta_f(S_t, t)}{B_t}$$

BSM PDE

BSM p.d.e. at the random point S_t.

$$\frac{1}{2}\sigma^2 S_t^2 f_{xx}(S_t, t) + (r - \delta) S_t f_x(S_t, t) + f_t(S_t, t) - r f(S_t, t) = 0$$

Deterministic BSM p.d.e..

$$\frac{1}{2}\sigma^2 x^2 f_{xx}(x, t) + (r - \delta) x f_x(x, t) + f_t(x, t) - r f(x, t) = 0$$

The physical expected return m cancels in the derivation. The BSM p.d.e. depends on r, δ, and σ, but not on m. This reflects no-arbitrage pricing by replication.

Final Condition

Final condition for terminal payoff Φ.

$$f(x, T) = \Phi(x)$$