UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in:	STK-MAT3710/4710 — Probability Theory
Day of examination:	8th December 2023
Examination hours:	9:00 am – 1:00 pm
This problem set consists of 2 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (2 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, X a random variable on it and $\mathcal{F} \subseteq \mathcal{A}$ a sub-σ-algebra.

(a) Define the concept of expectation of a random variable X from the measure-theoretical definition and outline the steps to construct it.

(b) Assuming that X is integrable, define the concept of conditional expectation of X given $\mathcal{F}$ through the theoretical definition.

Problem 2 Random variables (3 points)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent, identically distributed random variables with exponential distribution with parameter 1, i.e. each X_n is absolutely continuous with density function $f_{X_n}(x) = e^{-x}$, $x > 0$.

Define, for each $n \geq 1$, the events

$$A_n = \left\{ |X_n - X_{n-1}| \leq \frac{1}{n^2} \right\},$$

where $X_0 = 0$ by convention.

We call outcomes in A_n *friends*. In other words, we say that $X_{n-1}(\omega)$ and $X_n(\omega)$ are friends if $\omega \in A_n$.

(Continued on page 2.)

(a) Show that the random variables $Y_n = |X_n - X_{n-1}|$, $n \geq 1$ are again exponentially distributed with parameter 1. Hint: You may want to use $\mathbb{P}[A] = \int_{-\infty}^{\infty} \mathbb{P}[A|X = x] f_X(x) dx$ for an absolutely continuous random variable X with density f_X and any event A.

(b) Let I_n be the random variable that is one if X_{n-1} and X_n are friends and 0 otherwise. Show that $\{I_n\}_{n=1}^{\infty}$ converges in probability to 0.

(c) Will you observe an infinite number of friends in the infinite sequence $X_1, X_2, \ldots$? Justify your answer.

Problem 3 Conditional expectation (3 points)

Suppose that $a, b > 0$ real and X and Y are two random variables with values in $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{N}_0 = \{0, 1, \ldots\}$, respectively, whose distribution function is given by

$$\mathbb{P}[X \leq x, Y = n] = b \int_0^x \frac{(az)^n}{n!} e^{-(a+b)z} dz, \quad (x, n) \in \mathbb{R}_+ \times \mathbb{N}_0.$$

You may need the fact that $\int_0^{\infty} \frac{b^{n+1}}{n!} x^n e^{-bx} dx = 1$ and the Taylor expansion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{R}$. Also $\sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2}$ for $|r| < 1$.

(a) Prove that the marginal distributions of X and Y are given by

$$\mathbb{P}[X \leq x] = 1 - e^{-bx}, x \in \mathbb{R}_+, \quad \mathbb{P}[Y = n] = \frac{b}{a+b} \left(\frac{a}{a+b} \right)^n, n \in \mathbb{N}_0.$$

(b) Show that $\mathbb{E}[h(X)|Y = n] = \frac{(a+b)^{n+1}}{n!} \int_0^{\infty} h(x) x^n e^{-(a+b)x} dx$ for a Borel measurable function h.

(c) Compute $\mathbb{E}[X|\sigma(Y)]$ and verify that $\mathbb{E}[\mathbb{E}[X|\sigma(Y)]] = \mathbb{E}[X]$.

Problem 4 Martingales (2 points)

Let $\{M_n\}_{n=0}^{\infty}$ be a martingale relative to a filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$ and $\{X_n\}_{n=0}^{\infty}$ a bounded (i.e. $|X_n| \leq C$ for some constant $C > 0$ for all $n \geq 0$) stochastic process adapted to $\{\mathcal{F}_n\}_{n=0}^{\infty}$. Define the stochastic process Y_n, $n \geq 0$ by

$$Y_0 = 0, \quad Y_n = \sum_{k=1}^{n} X_{k-1}(M_k - M_{k-1}), \quad n \geq 1.$$

The process $\{Y_n\}_{n=0}^{\infty}$ is known as *discrete stochastic integral of* $\{X_n\}_{n=0}^{\infty}$ *with respect to the martingale* $\{M_n\}_{n=0}^{\infty}$.

(a) Show that $\{Y_n\}_{n=0}^{\infty}$ is a martingale relative to $\{\mathcal{F}_n\}_{n=0}^{\infty}$.

(b) Show that if, in addition, $\mathbb{E}[|M_n|^2] < \infty$ for every $n \geq 1$ then $\mathbb{E}[|Y_n^*|^2] < \infty$ for every $n \geq 1$ as well, where $Y_n^* = \sup_{k=0,\ldots,n} |Y_k|$.

GOOD LUCK!