UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Trial exam in:	STK3710/4710 — Probability Theory
Day of examination:	20th November 2023
Examination hours:	2:00 pm – 6:00 pm
This problem set consists of 11 pages.	
Appendices:	None
Permitted aids:	Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more oriented to solving through reasoning. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (3 points)

Let E be a set and $(F, \mathcal{F})$ a measurable space. Consider a transformation $f : E \to F$ between E and F. It is implicitly assumed that $f^{-1}(F) = E$.

(a) Given subsets $A, B \in \mathcal{F}$ show that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

and

$$f^{-1}(A^c) = f^{-1}(A)^c,$$

where $A^c = F \setminus A$ stands for the *complement of* A.

Now we show that $f^{-1}(A \cup B) \supseteq f^{-1}(A) \cup f^{-1}(B)$: let $x \in f^{-1}(A) \cup f^{-1}(B)$, then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. In particular, we can say that $x \in f^{-1}(A \cup B)$ or $x \in f^{-1}(A \cup B)$ and hence $x \in f^{-1}(A \cup B)$. Finally, observe that

$$E = f^{-1}(F) = f^{-1}(A \cup A^c) = f^{-1}(A) \cup f^{-1}(A^c),$$

where the last equality follows from the previous assertion. Then

$$f^{-1}(A)^c = E \setminus f^{-1}(A) = f^{-1}(A^c).$$

Points: Give 0.5 per each statement. Give 0.1 if the candidate tries to solve the exercise elementwise, but fails. Subtract 0.2 per mistake and 0.5 per serious mistakes.

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(b) Define a class of subsets of E as

$$\mathcal{E} = \{A \subseteq E : \text{ there exists } B \in \mathcal{F} \text{ with } A = f^{-1}(B)\}.$$

Show that $\mathcal{E}$ is a σ-algebra on E and that f is an $(\mathcal{E}, \mathcal{F})$-measurable transformation.

Solution: To show that $\mathcal{E}$ is a σ-algebra we need to show that the three axioms of σ-algebra are fulfilled. Namely,

1. $\emptyset \in \mathcal{E}$.
2. For a countable family of measurable sets $\{A_i\}_{i=1}^{\infty} \subset \mathcal{E}$ we have that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{E}$.
3. For every $A \in \mathcal{E}$ we have $A^c = E \setminus A \in \mathcal{E}$.

1. We have that $f^{-1}(\emptyset) = \{x \in E : f(x) \in \emptyset\} = \emptyset$ so, $\emptyset \in \mathcal{E}$.
2. Let now $\{A_i\}_{i=1}^{\infty} \subset \mathcal{E}$ be a countable family of measurable sets in $\mathcal{E}$. Since $A_i \in \mathcal{E}$ it means that there is an $B_i \in \mathcal{F}$ such that $A_i = f^{-1}(B_i)$. Hence,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} f^{-1}(B_i) = f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right).$$

Now, since $\mathcal{F}$ is a σ-algebra then $\bigcup_{i=1}^{\infty} B_i \in \mathcal{F}$. So, for $\bigcup_{i=1}^{\infty} A_i$ there is a set in $\mathcal{F}$, namely $\bigcup_{i=1}^{\infty} B_i$ such that $\bigcup_{i=1}^{\infty} A_i = f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)$, hence $\bigcup_{i=1}^{\infty} A_i \in \mathcal{E}$.
3. Finally, if $A \in \mathcal{E}$ then there is $B \in \mathcal{F}$ such that $A = f^{-1}(B)$. Then $A^c = f^{-1}(B)^c = f^{-1}(B^c)$ and since $B \in \mathcal{F}$ so is $B^c \in \mathcal{F}$ too. Hence, for A^c there is an B^c such that $A^c = f^{-1}(B^c)$ and hence $A^c \in \mathcal{E}$.
We need to prove that $f : (E, \mathcal{E}) \to (F, \mathcal{F})$ is measurable, that is for all $B \in \mathcal{F}$ we have $f^{-1}(B) \in \mathcal{E}$. Indeed, $\mathcal{E}$ is defined such way, that is $\mathcal{E} = \{f^{-1}(B), B \in \mathcal{F}\}$. Actually, $\mathcal{E}$ is the minimal σ-algebra that makes f measurable.

Points: Give 0.75 for the σ-algebra and 0.25 for the measurability of f. Give 0.25 only for writing down what is a σ-algebra and a measurable function but nothing else. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

(c) Let μ be a measure on $(E, \mathcal{E})$ where $\mathcal{E}$ is given as in item (b). Define the push-forward measure of μ via f and show that it is a measure on $(F, \mathcal{F})$.

Solution: The push-forward measure is defined as

$$\begin{aligned} mu_f : &\mathcal{F} \to [0, \infty] \\ &B \mapsto \mu_f(B) = \mu[f^{-1}(B)], \end{aligned}$$

where $\mu : \mathcal{E} \to [0, \infty]$ is a measure on $(E, \mathcal{E})$. We see that f^{-1} maps sets from $\mathcal{F}$ into sets from $\mathcal{E}$ and then μ maps sets from $\mathcal{E}$ into a value

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$[0,\infty]$.
We need to check that μ_f satisfies the axioms for measure which are:

1. $\mu_f[\emptyset] = 0$.
2. For a countable family of disjoint measurable sets $\{B_i\}_{i=1}^{\infty} \subset \mathcal{F}$ we have that
$$\mu_f\left[\bigcup_{i=1}^{\infty} B_i\right] = \sum_{i=1}^{\infty} \mu_f[B_i]$$
.

1. We have $\mu_f[\emptyset] = \mu[f^{-1}(\emptyset)] = \mu[\emptyset] = 0$, where the last step follows since μ *is* a measure.
2. Let $\{B_i\}_{i=1}^{\infty} \subset \mathcal{F}$ disjoint. Then
$$\begin{aligned}\mu_f\left[\bigcup_{i=1}^{\infty} B_i\right] &= \mu\left[f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right] \\ &= \mu\left[\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right].\end{aligned}$$

Now, we claim that if C, D are two disjoint sets, then $f^{-1}(C), f^{-1}(D)$ are also disjoint. Indeed, assume not. Then there is $x \in f^{-1}(C) \cap f^{-1}(D)$ which means that $x \in f^{-1}(C)$ *and* $x \in f^{-1}(D)$. Therefore, it means that there is an $y \in C$ such that $y = f(x)$ and there is an $z \in D$ such that $z = f(x)$ which implies $y = z$, in particular, $y \in C$ and $y \in D$ contradicting that C and D are disjoint.
To finish we simply use the fact that μ is a measure:
$$\begin{aligned}\mu_f\left[\bigcup_{i=1}^{\infty} B_i\right] &= \mu\left[\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right] \\ &= \sum_{i=1}^{\infty} \mu[f^{-1}(B_i)] \\ &= \sum_{i=1}^{\infty} \mu_f[B_i].\end{aligned}$$

Points: Give 0.5 for the definition of push-forward measure and 0.5 for the measurability. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

Problem 2 Random variables (2 points)

(a) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with the following distribution
$$\mathbb{P}[X_n = -n] = \mathbb{P}[X_n = n] = \frac{1}{2\sqrt{n}}, \quad \mathbb{P}[X_n = 0] = 1 - \frac{1}{\sqrt{n}}, \quad n \geq 1.$$

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Discuss the convergence of the sequence $\{X_n\}_{n=1}^{\infty}$ in L^p for any $p \geq 1$, almost surely, in probability and in distribution.

Solution: We can see that the probabilities for $-n$ and n vanish and for 0 prevails. Hence, we have see that $\{X_n\}_{n=1}^{\infty}$ converges, in distribution, to the constant 0, i.e.

$$X_n \xrightarrow{d} 0.$$

Let us check convergence in probability. We need to check whether for every $\varepsilon > 0$ we have

$$\lim_n \mathbb{P}[|X_n| > \varepsilon] = 0.$$

Let $\varepsilon > 0$ be small enough, say $\varepsilon < 1$. Then the event $\{|X_n| > \varepsilon\}$ is the event that $\{X_n = -n\}$ or $\{X_n = n\}$ that is $\{|X_n| = n\}$. Thus

$$\mathbb{P}[|X_n| > \varepsilon] = \mathbb{P}[|X_n| = n] = \frac{1}{\sqrt{n}} \xrightarrow{n\to\infty} 0.$$

Hence,

$$X_n \xrightarrow{P} 0,$$

as well.
For almost sure convergence, we aim to use the equivalent definition that for all $\varepsilon > 0$

$$\mathbb{P}[\limsup_n \{|X_n| > \varepsilon\}] = 0.$$

We have some work done. We know that

$$\mathbb{P}[\{|X_n| > \varepsilon\}] = \frac{1}{\sqrt{n}}.$$

We have

$$\sum_{n=1}^{\infty} \mathbb{P}[|X_n| > \varepsilon] = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

which implies, since $\{X_n\}_{n=1}^{\infty}$ are independent, by the second Borel-Cantelli lemma that

$$\mathbb{P}[\limsup_n \{|X_n| > \varepsilon\}] = 1$$

and hence convergence almost surely does not hold.
Now, we need to discuss the limit of $\mathbb{E}[|X_n|^p]$ for any $p \geq 1$. We have

$$\mathbb{E}[|X_n|^p] = |n|^p \frac{1}{2\sqrt{n}} + |n|^p \frac{1}{2\sqrt{n}} = n^{p-\frac{1}{2}}$$

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which does not converge to 0 for any $p \geq 1$. Hence, X_n does not converge in any L^p sense for any $p \geq 1$.

Points: Give 0.25 per each type of convergence. Give 0.25 for only writing down all the definitions of convergence without being used. Consider positively mentioning the use of Borel-Cantelli lemma for the almost sure convergence. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

(b) Consider the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$ the Borel σ-algebra on $[0,1]$ and $\mathbb{P}$ the Lebesgue measure, i.e. $\mathbb{P}[(a,b]) = b - a$ for all intervals $(a,b] \subset [0,1]$.

Consider on this probability space the random variables

$$X_n(\omega) = \frac{n+1}{n}\omega + (1-\omega)^n, \quad n \geq 1$$

and

$$X(\omega) = \omega.$$

Show that $\{X_n\}_{n=1}^{\infty}$ converges, almost surely, to X and discuss whether it also converges in L^p, for $p \geq 1$. You may find useful the following inequality: for real numbers a, b and $p \geq 1$, $(|a| + |b|)^p \leq 2^{p-1}(|a|^p + |b|^p)$.

Solution: We need to prove that

$$\mathbb{P}[\lim_n X_n = X] = 1.$$

Let $\omega \in (0,1]$, then

$$\lim_n X_n(\omega) = \lim_n \frac{n+1}{n}\omega + \lim_n (1-\omega)^n = \omega,$$

where we used that $0 \leq 1 - \omega < 1$ for $\omega \in (0,1]$ hence $\lim_n (1-\omega)^n = 0$. Hence,

$$\lim_n X_n(\omega) = \omega$$

for $\omega \in (0,1]$. Since $\mathbb{P}[(0,1]] = 1$ we have that $X_n \to X$ almost surely. You may notice that for $\omega = 0$ we have $\lim_n X_n(0) = 1$ and hence $X_n(0)$ does not converge to $X(0)$.
Now, let us study convergence in L^p, $p \geq 1$. We have

$$\begin{aligned}\mathbb{E}[|X_n - X|^p] &= \int_0^1 \left|\frac{n+1}{n}\omega + (1-\omega)^n - \omega\right|^p d\omega \\ &= \int_0^1 \left|\frac{1}{n}\omega + (1-\omega)^n\right|^p d\omega.\end{aligned}$$

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Using now the proposed inequality we have

$$\begin{aligned}\mathbb{E}[|X_n - X|^p] &\leq \int_0^1 \frac{1}{n}\omega^p + (1-\omega)^{np} d\omega \\ &= \frac{2^{p-1}}{p+1}\frac{1}{n} + 2^{p-1}\frac{1}{np+1}\end{aligned}$$

which converges to 0 as $n \to \infty$ for all $p \geq 1$. Hence, $\{X_n\}$ converges to X in L^p for all $p \geq 1$.

Points: Give 0.5 per each type of convergence. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

Problem 3 Conditional expectation (3 points)

Let $n \geq 1$ be a fixed natural number and $\{X_i\}_{i=1}^n$ a family of independent and identically distributed random variables. Let $S_n = \sum_{i=1}^n X_i$ denote their sum. Furthermore, consider $\mathcal{F} = \sigma(S_n)$ the σ-algebra generated by the sum of the n random variables.

(a) Compute $\mathbb{E}[X_1|\mathcal{F}]$.

Solution: One has the following intuitive fact:

$$\mathbb{E}[X_1|\mathcal{F}] = \cdots = \mathbb{E}[X_n|\mathcal{F}].$$

Recall the following result (Exercise from Chapter 7):

If X and Y are two random variables, then there is a measurable function f such that $\mathbb{E}[X|Y] = f(Y)$ a.s. Moreover, if (X', Y') is identically distributed to (X, Y) then $\mathbb{E}[X'|Y'] = f(Y')$ for the same function f.

Proving the above result is not difficult: the first part is a Theorem 7.3.1 from the lecture notes. The second part is reduced to showing that $f(Y')$ is indeed the same as $\mathbb{E}[X'|Y']$ which is implied by showing that $\mathbb{E}[\mathbb{I}_A f(Y')] = \mathbb{E}[\mathbb{I}_A X']$ for all $A \in \sigma(Y')$.
In our case we have (X_1, S_n) is identically distributed as (X_2, S_n) and (X_3, S_n) and so on. So the result above applies and we are, now certainly, able to say that

$$\mathbb{E}[X_1|\sigma(S_n)] = \mathbb{E}[X_2|\sigma(S_n)] = \cdots = \mathbb{E}[X_n|\sigma(S_n)].$$

Now, we can see that

$$\sum_{i=1}^n \mathbb{E}[X_i|\mathcal{F}] = n\mathbb{E}[X_1|\mathcal{F}].$$

But on the other hand, it is also true that

$$\sum_{i=1}^n \mathbb{E}[X_i|\mathcal{F}] = \mathbb{E}[\sum_{i=1}^n X_i|\mathcal{F}] = \mathbb{E}[S_n|\sigma(S_n)] = S_n.$$

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In conclusion,

$$\mathbb{E}[X_1|\mathcal{F}] = \frac{S_n}{n},$$

meaning that the best guess for a given random observation X_i, knowing the sum of all, is the sample mean.

Points: Give 0.25 for writing down the definition of conditional expectation. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

(b) Let Y, Z be two independent standard normally distributed random variables. Let $a, b \in \mathbb{R}$. Find the conditions on a and b for the random variables $(1-a)Y + bZ$ and $aY - bZ$ to be independent.

Solution: Let $(U, V) = g(Y, Z)$, where $g(y, z) = ((1-a)y + bz, ay - bz)$, then $g^{-1}(u, v) = \left(u + v, \frac{a}{b}u - \frac{1-a}{b}v\right)$ and

$$|Dg^{-1}(u, v)| = \left| \begin{pmatrix} 1 & 1 \\ \frac{a}{b} & -\frac{1-a}{b} \end{pmatrix} \right| = \left| -\frac{1-a}{b} - \frac{a}{b} \right| = \frac{1}{|b|}.$$

The density of the transformed vector is

$$\begin{aligned} f_{(U,V)}(u, v) &= f_{(Y,Z)}\left(g^{-1}(u, v)\right) |Dg^{-1}(u, v)| \\ &= f_Y(u + v) f_Z\left(\frac{a}{b}u - \frac{1-a}{b}v\right) \frac{1}{|b|}, \end{aligned}$$

where f_Y and f_Z are the density functions of a standard normal distribution.
Now, the function $f_{(U,V)}(u, v)$ will factor in u and v if the cross term in the exponent of the normal densities cancel out. The exponent we get is

$$-\frac{1}{2}\left[(u + v)^2 + \left(\frac{a}{b}u - \frac{1-a}{b}v\right)^2\right].$$

The cross term is:

$$2uv - 2\frac{a}{b}u\frac{1-a}{b}v.$$

So we need to impose that

$$2uv - 2\frac{a}{b}u\frac{1-a}{b}v = 0$$

which yields

$$(1-a)a = b^2.$$

An alternative way is to resort to the fact that if (X, Y) is a bivariate normal distribution and X and Y are uncorrelated, then X and Y are independent. This property is far from being true in general, but it holds for the particular case of normal distributions. Hence, it is enough to fins a and b such that $\mathbb{E}[((1-a)Y + Z)(aY - bZ)] = 0$.

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Points: Give 0.1 for writing down the formula for transformation. Give 0.25 for transforming the random vector correctly. Give 0.25 for stating independence correctly. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

(c) Assume now, in addition to independence, that the distribution of the X_i, $i = 1, \ldots, n$ is the standard normal distribution, that is $X_i \sim N(0,1)$ for al $i = 1, \ldots, n$. Compute $\mathbb{E}[X_1^2|\mathcal{F}]$.

Solution: To compute $\mathbb{E}[X_1^2|\mathcal{F}]$, we can not proceed as for $\mathbb{E}[X_1|\mathcal{F}]$ from (a). But, we may note that $X_1 - \frac{1}{n}S_n$ and $\frac{1}{n}S_n$ are independent. Indeed,

$$X_1 - \frac{1}{n}S_n = \left(1 - \frac{1}{n}\right) X_1 - \frac{1}{n}\sum_{i=2}^{n} X_i$$

and

$$\frac{1}{n}S_n = \frac{1}{n}X_1 + \frac{1}{n}\sum_{i=2}^{n} X_i.$$

We have $X_1 \sim N(0,1)$ and $\frac{1}{n}\sum_{i=2}^{n} X_i \sim N(0, \frac{n-1}{n^2}) \sim \frac{\sqrt{n-1}}{n}Z$ with $Z \sim N(0,1)$. Choose $Y = X_1$ and $a = \frac{1}{n}$ and $b = \frac{\sqrt{n-1}}{n}$ which satisfies the condition of item (a). Therefore,

$$X_1 - \frac{1}{n}S_n = (1 - \frac{1}{n})Y - \frac{\sqrt{n-1}}{n}Z$$

and

$$\frac{1}{n}S_n = \frac{1}{n}Y + \frac{\sqrt{n-1}}{n}Z$$

are independent.
In virtue of that, we can proceed as follows:

$$\mathbb{E}[X_1^2|\mathcal{F}] = \mathbb{E}[\left(X_1 - \frac{1}{n}S_n + \frac{1}{n}S_n\right)^2 |\mathcal{F}].$$

Now, since $\frac{1}{n}S_n$ is $\sigma(S_n)$-measurable, we can *freeze* $\frac{1}{n}S_n$ and use the fact that $X_1 - \frac{1}{n}S_n$ is independent of $\frac{1}{n}S_n$. Then,

$$\begin{aligned}
\mathbb{E}[X_1^2|\mathcal{F}] &= \mathbb{E}[\left(X_1 - \frac{1}{n}S_n + y\right)^2 |\mathcal{F}]\Big|_{y=\frac{1}{n}S_n} \\
&= \mathbb{E}[\left(X_1 - \frac{1}{n}S_n + y\right)^2]\Big|_{y=\frac{1}{n}S_n} \\
&= \mathbb{E}[(X_1 - \overline{X}_n + y)^2]\Big|_{y=\overline{X}_n} \\
&= \mathbb{E}[(X_1 - \overline{X}_n)^2] + 2\overline{X}_n\mathbb{E}[X_1 - \overline{X}_n] + \overline{X}_n^2,
\end{aligned}$$

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where $\overline{X}_n = \frac{1}{n}S_n$ is the sample mean.
Note that $\mathbb{E}[X_1 - \overline{X}_n] = 0$ and that

$$X_1 - \overline{X}_n \sim N(0, \frac{n-1}{n}).$$

Therefore,

$$\mathbb{E}[X_1^2|\mathcal{F}] = \frac{n-1}{n} + \overline{X}_n^2.$$

Points: Give 0.5 for applying the previous item correctly and observing the independence of $X_1 - \overline{X}_n$ and $\overline{X}_n$. Have consideration in trying to compute the conditional expectation using its properties. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

Problem 4 Martingales (2 points)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables all with distribution $N(-\mu, 1)$, $\mu > 0$ real.

(a) Find the values of $a \in \mathbb{R}$ for which the stochastic process $Y_n = e^{a\sum_{i=1}^n X_i}$, $n \geq 1$, is a submartingale, a martingale and a supermartingale with respect to the natural filtration of $\{X_n\}_{n=1}^{\infty}$.

Solution: For martingality we need to check the three conditions:

1. (adaptedness) Y_n is $\mathcal{F}_n$-measurable for all $n \geq 1$.
2. (integrability) $\mathbb{E}[|Y_n|] < \infty$ for all $n \geq 1$.
3. (martingale property) For $n, m \geq 1$ with $m \geq n$ we have

$$\mathbb{E}[Y_m|\mathcal{F}_n] = Y_n.$$

1. The first condition is easy to verify since

$$Y_n = f(X_1, \ldots, X_n)$$

for a measurable function f, namely $f(x_1, \ldots, x_n) = e^{a\sum_{i=1}^n x_i}$, actually f is continuous, so also measurable, for any finite value of a.
2. $\mathbb{E}[|Y_n|] = \mathbb{E}[Y_n]$ since $Y_n > 0$ a.s. ans furthermore, since X_i are i.i.d. normally distributed then $a\sum_{i=1}^n X_i \sim N(-an\mu, a^2 n)$. Hence, $e^{a\sum_{i=1}^n X_i}$ is log-normally distributed with mean $-an\mu$ and variance $a^2 n$. Hence,

$$\mathbb{E}[Y_n] = e^{-an\mu + \frac{1}{2}a^2 n} < \infty$$

for every $n \geq 1$ and any finite value of a.
3. Finally, let us check the martingale condition for $m = n + 1$,

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since by the tower property it will hold for any $m \geq n$. Indeed, if $\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n$ then

$$\mathbb{E}[Y_m|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Y_m|\mathcal{F}_{m-1}]|\mathcal{F}_n] = \mathbb{E}[Y_{m-1}|\mathcal{F}_n] = \cdots = \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n,$$

where we used the tower property and $\mathcal{F}_n \subset \mathcal{F}_{m-1}$ for $m \geq n+1$. So,

$$\begin{aligned}\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[e^{a\sum_{i=1}^{n+1} X_i}|\mathcal{F}_n] &= \mathbb{E}[e^{a\sum_{i=1}^{n} X_i} e^{aX_{n+1}}|\mathcal{F}_n] \\ &= \mathbb{E}[Y_n e^{aX_{n+1}}|\mathcal{F}_n] \\ &= Y_n \mathbb{E}[e^{aX_{n+1}}|\mathcal{F}_n] \\ &= Y_n e^{-a\mu+\frac{1}{2}a^2}.\end{aligned}$$

Then $\{Y_n\}_{n=1}^{\infty}$ will be a submartingale if, and only if

$$e^{-a\mu+\frac{1}{2}a^2} > 1,$$

a martingale if, and only if

$$e^{-a\mu+\frac{1}{2}a^2} = 1,$$

and a supermartingale if, and ony if

$$e^{-a\mu+\frac{1}{2}a^2} < 1.$$

The roots of $-a\mu + \frac{1}{2}a^2$ are $a = 0$ and $a = 2\mu$. Then $-a\mu + \frac{1}{2}a^2 < 0$ for $a < 0$ and $a > 2\mu$ and $-a\mu + \frac{1}{2}a^2 > 0$ for $0 < a < 2\mu$. So, $\{Y_n\}_{n=1}^{\infty}$ is submartingale if $a < 0$ or $a > 2\mu$, a martingale if $a = 0$ or $a = 2\mu$ and a supermartingale if $0 < a < 2\mu$.

Points: Give 0.5 for verifying martingality and 0.5 for sub- and supermartingality. Give 0.25 for writing down the definition of conditional expectation and nothing else. Give 0.2 for verifying the two first properties and 0.6 for the martingale property. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

(b) Assume that $a = 2\mu$ (then $\{Y_n\}_{n=1}^{\infty}$ is martingale) and let $x > 0$ real. Define $S_n = \sum_{i=1}^{n} X_i$. Show that

$$\mathbb{P}\left[\sup_{n\geq 1} S_n > x\right] \leq e^{-2\mu x}.$$

Solution: Let $Y_n = e^{2\mu S_n}$, $n \geq 1$. Then by (a) $\{Y_n\}_{n=1}^{\infty}$ is a martingale. This means that, by Doob's first martingale inequality we have

$$\mathbb{P}[Y_n^* > \alpha] \leq \frac{\mathbb{E}[Y_n]}{\alpha} = \frac{1}{\alpha},$$

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where $Y_n^* \triangleq \sup_{k=1,\dots,n} |Y_k|$.
However, this is not quite what we want, we wish an estimate for $\sup_n S_n$. Observe that

$$Y_n^* = \sup_{k=1,\dots,n} |Y_k| = \sup_{k=1,\dots,n} e^{2\mu S_k} = e^{2\mu \sup_{k=1,\dots,n} S_k}.$$

Therefore,

$$\log(Y_n^*) = 2\mu \sup_{k=1,\dots,n} S_k.$$

So,

$$\sup_n S_n = \frac{1}{2\mu} \lim_n \log(Y_n^*).$$

We turn now to estimating the probability. We have,

$$\begin{aligned}
\mathbb{P}\left[\sup_n S_n > x\right] &= \mathbb{P}\left[\frac{1}{2\mu} \lim_n \log(Y_n^*) > x\right] \\
&= \mathbb{P}\left[\log(\lim_n Y_n^*) > 2\mu x\right] \\
&= \mathbb{P}\left[\lim_n Y_n^* > e^{2\mu x}\right] \\
&\leq \limsup_n \mathbb{P}\left[Y_n^* > e^{2\mu x}\right] \\
&\leq \limsup_n \frac{\mathbb{E}[Y_n]}{e^{2\mu x}} \\
&= \limsup_n \frac{1}{e^{2\mu x}} \\
&= e^{-2\mu x},
\end{aligned}$$

where we used Doob's first martingale inequality from item (a) in the fifth equality.

Points: Give 0.2 for stating Doob's first martingale inequality. Give 0.2 for mentioning Fatou or subtract 0.2 if it is not mentioned. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

GOOD LUCK!

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