UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in:	STK-MAT3710/4710 — Probability Theory
Day of examination:	8th December 2023
Examination hours:	9:00 am – 1:00 pm
This problem set consists of 9 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Grading: The total score is 10 points. The grading scale is F [0,4), E [4,5), D [5, 6), C [6, 7), B [7, 8.5), A [8.5,10].

Problem 1 Theory (2 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, X a random variable on it and $\mathcal{F} \subseteq \mathcal{A}$ a sub-σ-algebra.

(a) Define the concept of expectation of a random variable X from the measure-theoretical definition and outline the steps to construct it.

Solution: The expectation of a random variable X is defined as the integral of the measurable function X w.r.t. the probability measure $\mathbb{P}$, that is

$$\mathbb{E}[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega).$$

The above quantity is constructed as follows: first you assume that X is a simple random variable, i.e. for events in the σ-algebra $\{A_i\}_{i=1}^n \subset \mathcal{A}$ and non-negative real numbers $\{a_i\}_{i=1}^n \subset [0, \infty)$ we have

$$X(\omega) = \sum_{i=1}^n a_i \mathbb{I}_{A_i}(\omega).$$

Then we define the integral as

$$\mathbb{E}[X] = \sum_{i=1}^n a_i \mathbb{P}[A_i].$$

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For any general non-negative random variable X, we have a result saying that there exists a increasing sequence of simple random variables $\{X_n\}_{n=1}^{\infty}$ such that $X_n \to X$ a.s. (or pointwise), then we define the integral of X as

$$\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n].$$

Finally, for any general random variable X, we split X as $X = X^+ - X^-$ where $X^+ = \max\{X, 0\}$ and $X^- = -\min\{X, 0\}$ the positive and negative parts which are non-negative and define

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-].$$

We say that X is integrable if, and only if

$$\mathbb{E}[|X|] = \mathbb{E}[X^+] + \mathbb{E}[X^-] < \infty.$$

Points: Give 0.25p if the expectation is defined for simple r.v.'s and 0.5p for non-negative r.v.'s via limit. Give the left 0.25p for a general r.v. splitting into positive and negative part. Give consideration if integrability of X is discussed or if, in addition, the connection to the push-forward measure is discussed. Give only 0.5p if the expectation is defined via the distribution/density functions. Subtract 0.2 per mistake and 0.5 per serious mistake.

(b) Assuming that X is integrable, define the concept of conditional expectation of X given $\mathcal{F}$ through the theoretical definition.

Solution: We define the *conditional expectation* of X, given $\mathcal{F}$ as the $\mathbb{P}$-a.s. unique $\mathbb{P}$-integrable and $\mathcal{F}$-measurable random variable Z such that

$$\int_F Z(\omega) d\mathbb{P}(\omega) = \int_F X(\omega) d\mathbb{P}(\omega)$$

for all events $F \in \mathcal{F}$. In such case, we denote Z by $\mathbb{E}[X|\mathcal{F}]$.
Points: Give 1p for everything. Subtract 0.5 if $\mathcal{F}$-measurability of Z is forgotten in the definition. Give max 0.25 if the candidate only provides the definition of $\mathbb{E}[X|F]$ for an event $F \in \mathcal{F}$. Subtract 0.2 per mistake and 0.5 per serious mistake.

Problem 2 Random variables (3 points)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent, identically distributed random variables with exponential distribution with parameter 1, i.e. each X_n is absolutely continuous with density function $f_{X_n}(x) = e^{-x}$, $x > 0$.

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Define, for each $n \geq 1$, the events

$$A_n = \left\{ |X_n - X_{n-1}| \leq \frac{1}{n^2} \right\},$$

where $X_0 = 0$ by convention.

We call outcomes in A_n *friends*. In other words, we say that $X_{n-1}(\omega)$ and $X_n(\omega)$ are friends if $\omega \in A_n$.

(a) Show that the random variables $Y_n = |X_n - X_{n-1}|$, $n \geq 1$ are again exponentially distributed with parameter 1. Hint: You may use the law of total probability for absolutely continuous random variables, that is $\mathbb{P}[A] = \int_{-\infty}^{\infty} \mathbb{P}[A|X = x] f_X(x) dx$ for an absolutely continuous random variable X with density f_X and any event A.

Solution: We apply the law of total expectation: let $x > 0$, then

$$\begin{aligned}
\mathbb{P}[|X_n - X_{n-1}| \leq x] &= \int_{-\infty}^{\infty} \mathbb{P}[|X_n - X_{n-1}| \leq x | X_{n-1} = y] f_{X_{n-1}}(y) dy \\
&= \int_0^{\infty} \mathbb{P}[|X_n - y| \leq x] e^{-y} dy \\
&= \int_0^{\infty} \mathbb{P}[-x \leq X_n - y \leq x] e^{-y} dy \\
&= \int_0^{\infty} \mathbb{P}[y - x \leq X_n \leq y + x] e^{-y} dy \\
&= \int_0^{\infty} \left(\mathbb{P}[X_n \leq y + x] - \mathbb{P}[X_n \leq y - x] \right) e^{-y} dy \\
&= \int_0^{\infty} \mathbb{P}[X_n \leq y + x] e^{-y} dy - \int_0^{\infty} \mathbb{P}[X_n \leq y - x] e^{-y} dy
\end{aligned}$$

Now, observe that the probability $\mathbb{P}[X_n \leq y - x]$ is 0 for $y < x$, so it makes only sense to integrate for $y > x$. Thus,

$$\begin{aligned}
&\mathbb{P}[|X_n - X_{n-1}| \leq x] = \\
&\qquad = \int_0^{\infty} \mathbb{P}[X_n \leq y + x] e^{-y} dy - \int_x^{\infty} \mathbb{P}[X_n \leq y - x] e^{-y} dy \\
&\qquad = \int_0^{\infty} \left(1 - e^{-(y+x)} \right) e^{-y} dy - \int_x^{\infty} \left(1 - e^{-(y-x)} \right) e^{-y} dy \\
&\qquad = 1 - \frac{e^{-x}}{2} - e^{-x} + \frac{e^{-x}}{2} \\
&\qquad = 1 - e^{-x}.
\end{aligned}$$

Hence, $\mathbb{P}[|X_n - X_{n-1}| \leq x]$, $x > 0$ which is the distribution function of an exponential random variable with parameter 1.

Points: Give full score if the candidate uses the transformation of variables formula or any other method (e.g. characteristic functions is possible too). Give 0.5 if there is a meaningful attempt, e.g. if the first display in the solution is written down. Subtract 0.2 per mistake and 0.5 per serious mistake.

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(b) Let I_n be the random variable that is one if X_{n-1} and X_n are friends and 0 otherwise. Show that $\{I_n\}_{n=1}^{\infty}$ converges in probability to 0.

Solution: The random variable I_n is defined as

$$I_n(\omega) = \begin{cases} 1, \text{ if } |X_n(\omega) - X_{n-1}(\omega)| \leq \frac{1}{n^2}, \\ 0, \text{ otherwise} \end{cases} = \begin{cases} 1, \text{ if } \omega \in A_n, \\ 0, \text{ otherwise} \end{cases}.$$

That is $I_n = \mathbb{I}_{A_n}$.
$\{I_n\}_{n\geq 1}$ will converge to 0 in probability if, and only if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}[|I_n| > \varepsilon] = 0.$$

Now, take $\varepsilon \in (0,1)$, then the event $\{|I_n| > \varepsilon\} = \{I_n = 1\}$. Therefore,

$$\mathbb{P}[|I_n| > \varepsilon] = \mathbb{P}[I_n = 1] = \mathbb{P}[A_n] = \mathbb{P}[|X_n - X_{n-1}| \leq \frac{1}{n^2}] = 1 - e^{-\frac{1}{n^2}},$$

where the last follows from item (a). Then, since $\lim_n e^{-\frac{1}{n^2}} = 1$ it follows that

$$\lim_n \mathbb{P}[|I_n| > \varepsilon] = \lim_n \left(1 - e^{-\frac{1}{n^2}}\right) = 0$$

and convergence in probability follows.
Another way of proving convergence is by simply observing that

$$\mathbb{P}[I_n = 1] = \mathbb{P}[A_n], \quad \mathbb{P}[I_n = 0] = 1 - \mathbb{P}[A_n]$$

and showing that $\lim_n \mathbb{P}[A_n] = 0$. This implies that I_n converges to 0 in distribution and since 0 is a constant then convergence in probability follows, as well.

Points: Give 0.25 for just writing the definition of convergence in probability. Give 0.25 extra for a meaningful attempt. Subtract 0.2 per mistake and 0.5 per serious mistake.

(c) Will you observe an infinite number of friends in the infinite sequence $X_1, X_2, \ldots$? Justify your answer.

Solution: The event that we observe infinitely many friends is given by $\limsup_n A_n$. Hence, we will observe infinitely many friends if $\mathbb{P}[\limsup_n A_n] = 1$ or not if $\mathbb{P}[\limsup_n A_n] = 0$. We may attempt to apply Borel-Cantelli's first lemma, since we do not need to assume that A_n are independent, which they are not. Then we have

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] = \sum_{n=1}^{\infty} \left(1 - e^{-\frac{1}{n^2}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

where we used the elementary estimate $1 - e^{-x} \leq x$ for $x > 0$. By Borel-Cantelli's first lemma we conclude that

$$\mathbb{P}[\limsup_n A_n] = 0$$

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and hence we will not observe an infinite number of friends.

Points: Give 0.5 for just writing Borel-Cantelli's lemma correct. Give 0.25 for mentioning which Borel-Cantelli lemma applies here. Subtract 0.2 per mistake and 0.5 per serious mistake.

Problem 3 Conditional expectation (3 points)

Suppose that $a, b > 0$ real and X and Y are two random variables with values in $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{N}_0 = \{0, 1, \dots\}$, respectively, whose distribution function is given by

$$\mathbb{P}[X \le x, Y = n] = b \int_0^x \frac{(az)^n}{n!} e^{-(a+b)z} dz, \quad (x, n) \in \mathbb{R}_+ \times \mathbb{N}_0.$$

You may need the fact that $\int_0^\infty \frac{b^{n+1}}{n!} x^n e^{-bx} dx = 1$ and the Taylor expansion $e^z = \sum_{n=0}^\infty \frac{z^n}{n!}$, $z \in \mathbb{R}$. Also $\sum_{n=0}^\infty n r^n = \frac{r}{(1-r)^2}$ for $|r| < 1$.

(a) Prove that the marginal distributions of X and Y are given by

$$\mathbb{P}[X \le x] = 1 - e^{-bx}, x \in \mathbb{R}_+, \quad \mathbb{P}[Y = n] = \frac{b}{a+b} \left(\frac{a}{a+b} \right)^n, n \in \mathbb{N}_0.$$

Solution: The marginal distributions are found by *integrating away* the other variable. Since Y takes values on $\mathbb{N}_+$ then

$$\mathbb{P}[X \le x] = \sum_{n=0}^\infty \mathbb{P}[X \le x, Y = n].$$

Then

$$\sum_{n=0}^\infty b \int_0^x \frac{(az)^n}{n!} e^{-(a+b)z} dz = b \lim_N \int_0^x \sum_{n=0}^N \frac{(az)^n}{n!} e^{-(a+b)z} dz.$$

In order to pull the limit inside, we note that the sequence of functions

$$f_N(z) = \sum_{n=0}^N \frac{(az)^n}{n!} e^{-(a+b)z}, \quad N \ge 0$$

is strictly increasing and converges pointwise in z towards

$$\lim_N f_N(z) = e^{-bz},$$

which is integrable. Hence, by Lebesgue's monotone convergence theorem, we have

$$\mathbb{P}[X \le x] = b \int_0^x \sum_{n=0}^\infty \frac{(az)^n}{n!} e^{-(a+b)z} dz.$$

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and since $\sum_{n=0}^{\infty} \frac{(az)^n}{n!} = e^{az}$ we conclude that

$$\mathbb{P}[X \leq x] = b \int_0^x e^{-bz} dz = 1 - e^{-bz},$$

which corresponds to an exponential distribution with parameter $b > 0$.
Similarly,

$$\begin{aligned}\mathbb{P}[Y = n] &= \lim_{x \to \infty} \mathbb{P}[X \leq x, Y = n] \\ &= b \int_0^{\infty} \frac{(az)^n}{n!} e^{-(a+b)z} dz = \frac{b}{a+b} \left(\frac{a}{a+b} \right)^n,\end{aligned}$$

where we used that $\int_0^{\infty} \frac{z^n}{n!} e^{-(a+b)z} dz = \frac{1}{(a+b)^n}$ as suggested by the exercise.

Points: Give 0.5 for each distribution. Subtract 0.2 if monotone convergence theorem is not mentioned. Give 0.2 per each, i.e. 0.4 if the candidate mentions how to compute the marginals. Subtract 0.2 per mistake and 0.5 per serious mistake.

(b) Show that $\mathbb{E}[h(X)|Y = n] = \frac{(a+b)^{n+1}}{n!} \int_0^{\infty} h(x) x^n e^{-(a+b)x} dx$ for a Borel measurable function h.

Solution: We simply need the conditional density of $X|Y = n$ which we almost have, since

$$\mathbb{P}[X \leq x|Y = n] = \frac{\mathbb{P}[X \leq x, Y = n]}{\mathbb{P}[Y = n]}.$$

Hence, the density of X, given $Y = n$ is

$$f_{X|Y=n}(x|n) = \frac{d}{dx} \mathbb{P}[X \leq x|Y = n] = \frac{\frac{d}{dx} \mathbb{P}[X \leq x, Y = n]}{\mathbb{P}[Y = n]}.$$

By the fundamental theorem of calculus we have

$$\frac{d}{dx} \mathbb{P}[X \leq x, Y = n] = b \frac{d}{dx} \int_0^x \frac{(az)^n}{n!} e^{-(a+b)z} dz = b \frac{(ax)^n}{n!} e^{-(a+b)x}.$$

Thus,

$$f_{X|Y=n}(x|n) = b \frac{(ax)^n}{n!} e^{-(a+b)x} \frac{a+b}{b} \left(\frac{a+b}{a} \right)^n = \frac{(a+b)^{n+1}}{n!} x^n e^{-(a+b)x}.$$

Finally,

$$\begin{aligned}\mathbb{E}[h(X)|Y = n] &= \int_0^{\infty} h(x) f_{X|Y=n}(x|n) dx \\ &= \frac{(a+b)^{n+1}}{n!} \int_0^{\infty} h(x) x^n e^{-(a+b)x} dx,\end{aligned}$$

and the formula follows.

Points: Give 0.5 for a general formula without completing the computation. Subtract 0.2 per mistake and 0.5 per serious mistake.

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(c) Compute $\mathbb{E}[X|\sigma(Y)]$ and verify that $\mathbb{E}[\mathbb{E}[X|\sigma(Y)]] = \mathbb{E}[X]$.

Solution: To compute $\mathbb{E}[X|\sigma(Y)]$ we simply acknowledge that this is an $\sigma(Y)$-measurable random variable, i.e. a (Borel) measurable function of Y. We may just "pretend" that $Y = n$ and use the formula from (b) with $h(x) = x$. Hence,

$$\mathbb{E}[X|Y = n] = \frac{(a+b)^n}{n!} \int_0^\infty x^{n+1} e^{-(a+b)x} dx.$$

Then we use the formula suggested in the exercise to conclude that

$$\mathbb{E}[X|Y = n] = \frac{(a+b)^{n+1}}{n!} \int_0^\infty x^{n+1} e^{-(a+b)x} dx = \frac{n+1}{a+b}.$$

As a result,

$$\mathbb{E}[X|\sigma(Y)] = \frac{Y+1}{a+b},$$

which we can confirm is a Borel measurable function of Y. Now, we can compute $\mathbb{E}[X]$ by the tower property:

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X|\sigma(Y)]] \\ &= \mathbb{E}\left[\frac{Y+1}{a+b}\right] \\ &= \frac{\mathbb{E}[Y]+1}{a+b}.\end{aligned}$$

Now,

$$\mathbb{E}[Y] = \sum_{n=0}^\infty n\mathbb{P}[Y = n] = \frac{b}{a+b} \sum_{n=0}^\infty n \left(\frac{a}{a+b}\right)^n = \frac{a}{b},$$

by applying the formula $\sum_{n=0}^\infty nr^n = \frac{r}{(1-r)^2}$ with $r = a/(a+b)$. We conclude that

$$\mathbb{E}[X] = \frac{1}{b}.$$

If one recognizes in (a) that $X \sim \exp(b)$ then one can confirm that the result is correct and that the tower property obviously holds.

Points: Give 0.5 for each expectation, but only give points for $\mathbb{E}[X]$ if computed via tower property. Subtract 0.2 if $\mathbb{E}[X|\sigma(Y)]$ is not recognized as a measurable function of Y. Subtract 0.2 per mistake and 0.5 per serious mistake.

Problem 4 Martingales (2 points)

Let $\{M_n\}_{n=0}^\infty$ be a martingale relative to a filtration $\{\mathcal{F}_n\}_{n=0}^\infty$ and $\{X_n\}_{n=0}^\infty$ a bounded (i.e. $|X_n| \leq C$ for some constant $C > 0$ for all $n \geq 0$) stochastic process adapted to $\{\mathcal{F}_n\}_{n=0}^\infty$. Define the stochastic process Y_n, $n \geq 0$ by

$$Y_0 = 0, \quad Y_n = \sum_{k=1}^n X_{k-1}(M_k - M_{k-1}), \quad n \geq 1.$$

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The process $\{Y_n\}_{n=0}^{\infty}$ is known as *discrete stochastic integral of* $\{X_n\}_{n=0}^{\infty}$ *with respect to the martingale* $\{M_n\}_{n=0}^{\infty}$.

(a) Show that $\{Y_n\}_{n=0}^{\infty}$ is a martingale relative to $\{\mathcal{F}_n\}_{n=0}^{\infty}$.

Solution: We need to check the three conditions for martingale.

1. (Adaptedness): Y_n needs to be $\mathcal{F}_n$-measurable. Indeed,

$$Y_n = \sum_{k=1}^{n} X_{k-1}(M_k - M_{k-1}).$$

We have that X_{k-1} is $\mathcal{F}_{k-1}$-measurable and since $\mathcal{F}_{k-1} \subset \mathcal{F}_n$ for all $k = 1, \ldots, n$ then X_{k-1} is also $\mathcal{F}_n$-measurable. Also, $M_k - M_{k-1}$ are all $\mathcal{F}_k$-measurable for $k = 1, \ldots, n$ since M_k is a martingale. Since $\mathcal{F}_k \subset \mathcal{F}_n$ for all $k = 0, \ldots, n$. Hence, $M_k - M_{k-1}$ is also $\mathcal{F}_n$-measurable. Then $X_{k-1}(M_k - M_{k-1})$ is $\mathcal{F}_n$-measurable for all $k = 1, \ldots, n$ and hence the sum is also $\mathcal{F}_n$-measurable.

2. (Integrability): We need to check that $\mathbb{E}[|Y_n|] < \infty$ for all $n \geq 1$. By the triangle inequality we have

$$\mathbb{E}[|Y_n|] \leq \sum_{k=1}^{n} \mathbb{E}[|X_{k-1}||M_k - M_{k-1}|].$$

Since $|X_{k-1}| \leq C$ then

$$\mathbb{E}[|Y_n|] \leq C \sum_{k=1}^{n} \mathbb{E}[|M_k - M_{k-1}|].$$

Finally, since $\{M_n\}_{n=0}^{\infty}$ is a martingale we have $\mathbb{E}[|M_n|] < \infty$ for all $n \geq 1$, hence $\mathbb{E}[|M_k - M_{k-1}|] \leq \mathbb{E}[|M_k|] + \mathbb{E}[|M_{k-1}|] < \infty$ for all $k = 1, \ldots, n$.

3. (Martingale property): It suffices to check

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n, \quad n \geq 0,$$

since by the tower property it holds for $m \geq n$,

$$\mathbb{E}[Y_m|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Y_m|\mathcal{F}_{m-1}]|\mathcal{F}_n] = \cdots = \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n.$$

Then,

$$\begin{aligned}\mathbb{E}[Y_{n+1}|\mathcal{F}_n] &= \mathbb{E}[Y_n + X_n(M_{n+1} - M_n)|\mathcal{F}_n] \\ &= Y_n + \mathbb{E}[X_n(M_{n+1} - M_n)|\mathcal{F}_n],\end{aligned}$$

where we used that Y_n is $\mathcal{F}_n$-measurable. Now, since $\{X_n\}_{n=1}^{\infty}$ is adapted, we have that X_n is $\mathcal{F}_n$-measurable, hence we can take

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it out of the conditional expectation. Finally, since $\{M_n\}_{n=0}^{\infty}$ is a martingale, we have

$$\mathbb{E}[M_{n+1} - M_n|\mathcal{F}_n] = \mathbb{E}[M_{n+1}|\mathcal{F}_n] - \mathbb{E}[M_n|\mathcal{F}_n] = M_n - M_n = 0$$

and hence,

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n,$$

for every $n \geq 0$ and the result follows.

Points: Give 0.5 for checking adaptedness and integrability and 0.5 for the martingale property. For not mentioning properties of conditional expectation, subtract 0.2 per each (e.g. measurability, towerproperty, etc.). Subtract 0.2 per mistake and 0.5 per serious mistake.

(b) Show that if, in addition, $\mathbb{E}[|M_n|^2] < \infty$ for every $n \geq 1$ then $\mathbb{E}[|Y_n^*|^2] < \infty$ for every $n \geq 1$ as well, where $Y_n^* = \sup_{k=0,\ldots,n} |Y_k|$.

Solution: Since $\{Y_n\}_{n=1}^{\infty}$ is a martingale by item (i), then by Doob's L^p martingale inequality with $p = 2$ we have

$$\mathbb{E}[|Y_n^*|^2] \leq K\mathbb{E}[|Y_n|^2]$$

for a constant $K > 0$ that is not relevant (actually $K = 4$). Observe that

$$|Y_n|^2 \leq n \sum_{k=1}^{n} |X_{k-1}|^2 |M_k - M_{k-1}|^2$$

and since $|X_{k-1}| \leq C$ so $|X_{k-1}|^2 \leq C^2$. Furthermore,

$$\mathbb{E}[|Y_n|^2] \leq C^2 n \sum_{k=1}^{n} \mathbb{E}[|M_k - M_{k-1}|^2] < \infty$$

for each $n \geq 1$ since $\mathbb{E}[|M_k|^2] < \infty$ for each $k = 0, \ldots, n$. Altogether,

$$\mathbb{E}[|Y_n^*|^2] \leq KC^2 n \sum_{k=1}^{n} \mathbb{E}[|M_k - M_{k-1}|^2] < \infty$$

for each $n \geq 1$.

Points: Give 0.25 for only mentioning Doob's L^p martingale inequality. Give 0.5 if it is in addition applied and give 0.75 if the candidate uses the boundedness of X_k but it remains to conclude. Subtract 0.2 per mistake and 0.5 per serious mistake.

GOOD LUCK!