UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK-MAT3710/4710 — Probability Theory
Day of examination:	16th December 2024
Examination hours:	3:00 pm – 7:00 pm
This problem set consists of 2 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature and the three remaining ones are more applied. All items within a problem are worth the same unless otherwise specified. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (2 points)

Let E be an uncountable set. Define the class of subsets of E given by

$$\mathcal{E} = \{A \subseteq E : \text{ either } A \text{ is countable, or } A^c \text{ countable}\}.$$

(a) Show that if $A, B \subseteq E$ are uncountable such that $A \cap B = \emptyset$, then either $A \notin \mathcal{E}$ or $B \notin \mathcal{E}$.

(b) Show that $\mathcal{E}$ is a σ-algebra.

(c) Define the mapping $\mu : \mathcal{E} \to [0, \infty]$ by $\mu(X) = \mathbb{I}_{\{X^c \text{ countable}\}}$. Show that μ is a probability measure.

Problem 2 Random variables (3 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\{X_n\}_{n=1}^{\infty}$ a sequence of independent identically distributed random variables with uniform distribution on $[0,1]$, i.e. with densities $f_{X_n}(x) = 1$, $x \in [0,1]$, $n \geq 1$.

Define the random variables

$$L_{(1:n)} = \min\{X_1, \ldots, X_n\}, \quad H_{(1:n)} = \max\{X_1, \ldots, X_n\}, \quad n \geq 2.$$

It may not be surprising that $L_{(1:n)}$ and $H_{(1:n)}$ are not independent and, in fact, it can be shown that their joint density is given by

$$f_{(L_{(1:n)}, H_{(1:n)})}(l, h) = n(n-1)(h-l)^{n-2}, \quad 0 \leq l < h \leq 1.$$

(Continued on page 2.)

(a) Prove that the density function of the range, denoted by R_n, of n independent uniform observations is given by

$$f_{R_n}(r) = n(n-1)r^{n-2}(1-r), \quad 0 < r < 1.$$

(b) Study the convergence of the sequence $\{R_n\}_{n=1}^{\infty}$ in distribution, probability, almost surely and in L^p, $p \geq 1$. Note: $\sum_{n=1}^{\infty} nr^n < \infty$ for every real r such that $|r| < 1$.

(c) Consider independent observations $X_1, X_2, \ldots$ of the uniform distribution. What is the probability that we observe k consecutive observations that are, at most, within distance $\delta > 0$?

Problem 3 Conditional expectation (2 points)

Let X and Y be two independent Bernoulli distributed random variables in some probability space, i.e. $X, Y \in \{0, 1\}$ and $\mathbb{P}[X = 1] = \mathbb{P}[Y = 1] = p$ for some $p \in [0, 1]$. Define the random variable $Z = \mathbb{I}_{\{X+Y=0\}}$.

(a) Compute $\mathbb{E}[Z]$, $\mathbb{E}[X|Z]$ and verify that $\mathbb{E}[\mathbb{E}[X|Z]] = \mathbb{E}[X]$.

(b) Compute $\mathbb{E}[Z|X]$. Is it true that $\mathbb{E}[\mathbb{E}[X|Z]|X] = \mathbb{E}[\mathbb{E}[Z|X]|Z]$? What about $\mathbb{E}[\mathbb{E}[X|Z]|X, Z] = \mathbb{E}[X|Z]$? Justify your answer.

Problem 4 Martingales (3 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and X, Y two independent random variables in it. Assume that X is uniformly distributed on $[0, 1]$, i.e. $f(x) = 1$ for $x \in [0, 1]$ and $\mathbb{P}[Y = -1] = \mathbb{P}[Y = 1] = \frac{1}{2}$. Consider the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(X)$ and $\mathcal{F}_n = \sigma(X, Y)$ for all $n \geq 2$. Define for each $n \geq 1$ the processes

$$M_n = \frac{Y}{X}\mathbb{I}_{\{n \geq 2\}}, \quad \tau_n = \begin{cases} 1, & n < \frac{1}{X}, \\ \infty, & n \geq \frac{1}{X} \end{cases}.$$

(a) Show that $\{M_n\}_{n \geq 1}$ is not a martingale.

(b) Show that $\{\tau_n\}_{n \geq 1}$ is a monotone sequence of stopping times which converges almost surely to infinity. Note: we have defined stopping times as being finite. Here, we allow infinity to be an outcome. The definition remains the same with the additional requirement that $\{\tau_n = \infty\}$ also needs to be measurable.

(c) Prove that $\{M_{\tau_k \wedge n}\}_{n \geq 1}$ is a martingale for every $k \geq 1$. Here, $x \wedge y \triangleq \min\{x, y\}$. A process like $\{M_n\}_{n \geq 1}$ is known as local martingale.

GOOD LUCK!