UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Trial exam in:	STK3710/4710 — Probability Theory
Day of examination:	20th November 2024
Examination hours:	2:00 pm – 6:00 pm
This problem set consists of 9 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (3 points)

Let $\{X_n\}_{n=1}^{\infty}$, X be random variables defined in a probability space.

(a) (1 point) Define the concept of convergence in probability and convergence almost surely.

Solution: We say that a sequence of random variables $\{X_n\}_{n\geq 1}$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ converges *in probability* to a random variable X on the same probability space if, and only if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\left[|X_n - X| > \varepsilon\right] = 0.$$

We say that a sequence of random variables $\{X_n\}_{n\geq 1}$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ converges *almost surely* to a random variable X on the same probability space if, and only if

$$\mathbb{P}\left[\lim_{n\to\infty} X_n = X\right] = 1.$$

Points: Give 0.5p per definition. Give 0.5 for equivalent definitions. Subtract 0.2 for mistakes.

(b) (2 points) Prove that $X_n \xrightarrow{n\to\infty} X$ in probability if, and only if

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{|X_n - X|}{1 + |X_n - X|}\right] = 0.$$

(Continued on page 2.)

Hint: Note that $\frac{|x|}{1+|x|} < 1$ for all x and that $f(x) = \frac{x}{1+x}$ is an increasing function for $x \geq 0$. It may also be useful to split the sample space into the events $\{|X_n - X| > \varepsilon\}$ and $\{|X_n - X| \leq \varepsilon\}$.

Solution: Note that X does not need to be integrable since $\frac{1}{1+|x|} \leq 1$ for any x and hence the expectations are always finite.

Let us prove that convergence in probability implies the given limit. We assume that for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\left[|X_n - X| > \varepsilon\right] = 0.$$

Then

$$\begin{aligned}\mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\right] &= \mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\mathbb{I}_{\{|X_n - X| \leq \varepsilon\}}\right] + \mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\mathbb{I}_{\{|X_n - X| > \varepsilon\}}\right] \\ &\leq \varepsilon + \mathbb{E}\left[\mathbb{I}_{\{|X_n - X| > \varepsilon\}}\right] \\ &= \varepsilon + \mathbb{P}\left[|X_n - X| > \varepsilon\right],\end{aligned}$$

where we used that $\frac{1}{1+|x|} \leq 1$ on the first term and that $\frac{|x|}{1+|x|} \leq 1$ on the second. Since the probability converges to zero by assumption it means that for every $\varepsilon > 0$ there is an n_0 such that for all $n \geq n_0$ $\mathbb{P}[|X_n - X| > \varepsilon] \leq \varepsilon$ and in particular for all $n \geq n_0$ we have

$$\mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\right] \leq 2\varepsilon,$$

hence the limit follows.

On the contrary, we have

$$\mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\right] \geq \mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\mathbb{I}_{\{|X_n - X| > \varepsilon\}}\right] \geq \frac{\varepsilon}{1+\varepsilon}\mathbb{P}\left[|X_n - X| > \varepsilon\right],$$

where we used that $f(x) = \frac{x}{1+x}$ is increasing for $x \geq 0$ and hence $f(|X_n - X|) \geq f(\varepsilon)$. Then since the left-hand side converges to zero and the right-hand side is positive. We have convergence to zero and hence convergence in probability.

Points: Give 1p for each implication. Within each implication, give 0.5 if there is a meaningful attempt and 0.25 if Markov inequality is mentioned. Subtract 0.2 for mistakes.

Problem 2 Random variables (2 points)

The Pareto distribution is an absolutely continuous probability distribution characterized by having the following density function,

$$f(x) = \frac{\alpha\lambda^\alpha}{x^{\alpha+1}}, \quad x \geq \lambda,$$

(Continued on page 3.)

for positive real parameters α, λ and $\alpha > 1$ to guarantee finite expectation. Note that the distribution function is given by $F(x) = 1 - \left(\frac{\lambda}{x}\right)^{\alpha}$, $x \geq \lambda$.

Consider $n \geq 1$ independent random variables $X_1, \ldots, X_n$ where X_i is Pareto-distributed with parameters $\lambda > 0$ and $\alpha_i > 1$, $i = 1, \ldots, n$.

Define the the random variables $Y_n = \min\{X_1, \ldots, X_n\}$, $n \geq 1$.

(a) Find the distribution of Y_n and its expectation.

Solution: Note that $Y_{(n)}$ takes values in $[\lambda, \infty)$. Let $y \geq \lambda$, then

$$\mathbb{P}[Y_n > y] = \mathbb{P}[X_1 > y, \ldots, X_n > y] = \prod_{i=1}^{n} \mathbb{P}[Y_i > y] = \prod_{i=1}^{n} \left(\frac{\lambda}{y}\right)^{\alpha_i} = \left(\frac{\lambda}{y}\right)^{\sum_{i=1}^{n} \alpha_i}$$

hence $Y_{(n)}$ is Pareto-distributed with parameters λ and $\sum_{i=1}^{n} \alpha_i$.

The mean is given by

$$\mathbb{E}[Y_{(n)}] = \int_{\lambda}^{\infty} x d f_n(x),$$

where $f_n(x) \triangleq \frac{a_n \lambda^{a_n}}{x^{a_n+1}}$ and $a_n = \sum_{i=1}^{n} \alpha_i$. Thus we have

$$\mathbb{E}[Y_{(n)}] = a_n \lambda^{a_n} \int_{\lambda}^{\infty} x^{-a_n} dx = \frac{\lambda a_n}{a_n - 1}.$$

Points: Give 0.5 for the distribution and 0.5 for the mean. Give 0.25 for finding the tail distribution. Give 0.1 for writing down an expression for the expectation. Subtract 0.2 for mistakes.

(b) Give a sufficient condition on the sequence of parameters $\{\alpha_i\}_{i=1}^{n}$ to guarantee that $\{Y_n\}_{n=1}^{\infty}$ converges almost surely to λ. Does the sequence converge in L^1 under your condition? Why or why not.

Solution: We try to apply Borel-Cantelli's first lemma to show that the sequence converges almost surely to λ. Define the events $A_n = \{|Y_{(n)} - \lambda| > \varepsilon\}$ for an arbitrarily small $\varepsilon > 0$. Denote by $a_n = \sum_{i=1}^{n} \alpha_i$. Then

$$\begin{aligned} \mathbb{P}[A_n] &= \mathbb{P}[|Y_{(n)} - \lambda| > \varepsilon] \\ &= \mathbb{P}[Y_{(n)} > \lambda + \varepsilon] \\ &= \left(\frac{\lambda}{\lambda + \varepsilon}\right)^{a_n}. \end{aligned}$$

A general sufficient condition would be

$$\sum_{n=1}^{\infty} r^{a_n} < \infty \tag{1}$$

(Continued on page 4.)

for $|r| < 1$ so that we have

$$\mathbb{P}[\limsup_n A_n] = 0$$

and hence $\{Y_{(n)}\}_{n=1}^{\infty}$ converges to λ a.s. An example is simply taking $\alpha_i = \alpha$ constant, then $a_n = \alpha n$.

Yes, the sequence converges in L^1 trivially since $Y_{(n)} \geq \lambda$ hence

$$\lim_n \mathbb{E}[|Y_{(n)} - \lambda|] = \lim_n \mathbb{E}[Y_{(n)}] - \lambda = \lim_n \frac{\lambda a_n}{a_n - 1} - \lambda = 0$$

provided $\lim_n a_n = \infty$, which is the case if condition (1) holds.

Points: Give 0.5 for mentioning and trying to apply the right Borel-Cantelli lemma. Give 0.5 for the L^1-convergence. Subtract 0.2 for mistakes.

Problem 3 Conditional expectation (3 points)

We consider an insurance portfolio in which there is a large number n of policyholders. We model the number of the insurance claims of the policyholders by i.i.d. random variables $X_1, \ldots, X_n$. Thus X_i is the number of claims of policyholder i. Let $p_k \triangleq \mathbb{P}[X_i = k]$ for all $i = 1, \ldots, n$ and denote by $m = \mathbb{E}[X_i] = \sum_{k\geq 1} k p_k < \infty$ and $s^2 = \mathbb{E}[X_i^2] - m^2 = \sum_{k\geq 1} k^2 p_k - m^2 < \infty$ the expectation and variance.

We pick an insurance claim uniformly at random. This claim belongs to some policyholder among these n. Denote by Z_n be the number of claims of such policyholder.

(a) Justify that for every integer $k \geq 1$ we have

$$\mathbb{P}[Z_n = k | X_1, \ldots, X_n] = \frac{k \sum_{i=1}^n \mathbb{I}_{\{X_i = k\}}}{\sum_{i=1}^n X_i}.$$

Solution: The portfolio has $X_1 + \cdots + X_n$ claims in total. Given a total number of claims $X_1 + \cdots + X_n$ we choose at random one. Then we want to see how many among all the n have exactly k claims. This number is $\mathbb{I}_{\{X_1=k\}} + \cdots + \mathbb{I}_{\{X_n=k\}}$. Then the total number of claims among all policyholders with exactly k claims is $k \sum_{i=1} \mathbb{I}_{\{X_i=k\}}$. Hence, the probability that we observe k claims in a randomly chosen claim from a policyholder among the n is given by

$$\mathbb{P}[Z_n = k | X_1, \ldots, X_n] = \frac{k \sum_{i=1}^n \mathbb{I}_{\{X_i = k\}}}{\sum_{i=1}^n X_i}.$$

Points: Give 1p for this item. Give 0.5 for some meaningful justification. Subtract 0.2 for mistakes.

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(b) Show that $\{Z_n\}_{n=1}^{\infty}$ converges in distribution to a random variable Z_∞ with probability mass function given by $\mathbb{P}[Z_\infty = k] = \frac{kp_k}{m}$, $k \geq 1$.

Solution: By dividing by n we have

$$\mathbb{P}[Z_n = k|X_1, \ldots, X_n] = \frac{k\frac{1}{n}\sum_{i=1}^n \mathbb{I}_{\{X_i=k\}}}{\frac{1}{n}\sum_{i=1}^n X_i}.$$

Since X_i are independent and identically distributed, we have, by the strong law of large numbers, that

$$\frac{1}{n}\sum_{i=1}^n \mathbb{I}_{\{X_i=k\}} \to \mathbb{E}[\mathbb{I}_{\{X_1=k\}}] = \mathbb{P}[X_1 = k] = p_k$$

and

$$\frac{1}{n}\sum_{i=1}^n X_i \to \mathbb{E}[X_1] = m.$$

Hence,

$$\mathbb{P}[Z_n = k|X_1, \ldots, X_n] \to \frac{kp_k}{m}.$$

Now,

$$\mathbb{P}[Z_n = k] = \mathbb{E}[\mathbb{P}[Z_n = k|X_1, \ldots, X_n]].$$

Since $\mathbb{P}[Z_n = k|X_1, \ldots, X_n]$ is a.s. bounded (by one) we have by Lebesgue's dominated convergence theorem that

$$\mathbb{P}[Z_n = k] = \mathbb{E}[\mathbb{P}[Z_n = k|X_1, \ldots, X_n]] \to \frac{kp_k}{m}$$

almost surely.

Points: Give 0.5 for detecting strong law of large numbers. Give 0.25 for mentioning dominated convergence (or subtract for forgetting). Subtract 0.2 for mistakes.

(c) Show that

$$\mathbb{E}[Z_n] \xrightarrow{n\to\infty} \frac{s^2 + m^2}{m}.$$

Solution:

Z_n converges in distribution to Z_∞ with the given distribution. Convergence in distribution does not imply convergence of the means to the limiting mean. In other words, $Z_n \to Z_\infty$ in distribution does not imply $\mathbb{E}[Z_n] \to \mathbb{E}[Z_\infty]$. Example: $\mathbb{P}[X_n = n] = \frac{1}{n}$ and $\mathbb{P}[X_n = 0] = 1 - \frac{1}{n}$. Then $X_n \to 0$ in distribution but $\mathbb{E}[X_n] = 1$ which is not $\mathbb{E}[X_\infty] = 0$. But in this case it is true! There are two ways we can expect convergence of the means when having «only» convergence in distribution. One is having convergence in mean, i.e. $\mathbb{E}[|Z_n - Z_\infty] \to 0$ which is stronger than convergence in probability

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and distribution and another one is having uniform integrability. Indeed,

$$\lim_{C\to\infty}\sup_n \mathbb{E}[|Z_n|\mathbb{I}_{\{|Z_n|\geq C\}}] = 0.$$

Note that

$$\begin{aligned}\mathbb{E}[|Z_n|\mathbb{I}_{\{|Z_n|\geq C\}}] &= \mathbb{E}\left[k\mathbb{I}_{\{k\geq C\}}\frac{k\sum_{i=1}^n \mathbb{I}_{\{X_i=k\}}}{\sum_{i=1}^n X_i}\right]\\ &= \mathbb{E}\left[\frac{\sum_{i=1}^n X_i^2\mathbb{I}_{\{X_i\geq C\}}}{\sum_{i=1}^n X_i}\right].\end{aligned}$$

By the strong law of large numbers we have

$$\frac{\frac{1}{n}\sum_{i=1}^n X_i^2\mathbb{I}_{\{X_i\geq C\}}}{\frac{1}{n}\sum_{i=1}^n X_i} \to \frac{\mathbb{E}[X_1^2\mathbb{I}_{\{X_1\geq C\}}]}{m}.$$

Since $\mathbb{E}[|Z_n|\mathbb{I}_{\{|Z_n|\geq C\}}]$ is a convergent sequence (strong law of large numbers) then it is bounded. Thus

$$\sup_n \mathbb{E}[|Z_n|\mathbb{I}_{\{|Z_n|\geq C\}}] < \infty.$$

Now, without loss of generality assume that C is integer, then

$$\mathbb{E}[X_1^2\mathbb{I}_{\{X_1\geq C\}}] = \sum_{k=C}^{\infty} k^2 p_k.$$

The last term is the tail of a convergent series, so it must converge to 0. In a summary,

$$\lim_{C\to\infty}\sup_n \mathbb{E}[|Z_n|\mathbb{I}_{\{|Z_n|\geq C\}}] = 0.$$

Therefore,

$$\mathbb{E}[Z_n] \to \mathbb{E}[Z_\infty] = \sum_k k\frac{kp_k}{m} = \frac{s^2+m^2}{m}.$$

Observation: Note that a sufficient condition for $\{Z_n\}_n$ to be uniformly integrable is by Theorem 8.7.3 from the lecture notes, that $\sup_n \mathbb{E}[|Z_n|^p] < \infty$ for some $p > 1$. But

$$\mathbb{E}[|Z_n|^p] = \mathbb{E}\left[\frac{\sum_{i=1}^n X_i^{p+1}}{\sum_{i=1}^n X_i}\right]$$

and even for p very close to 1, we have $1 + p > 2$ and we have not assumed that X_i has higher order moments than 2. Hence, this

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theorem fails and one needs to prove uniform integrability from the very definition! (Definition 8.7.1).

Finally, if you want, you can try to show that Z_n actually also converges to the limit $L = \frac{s^2+m^2}{m}$ in L^1. It is a nice exercise. Hint: you need to combine the law of total expectation as we have done several times, then get rid of the absolute value $|Z_n - L|$, by considering the cases that $Z_n > L$ and $Z_n < L$ and finally apply the strong law of large numbers.

Points: Give 0.2 for noting the law of total expectation. Give 0.5 for computing the conditional expectation. Give 0.25 for the law of large numbers and 0.25 for Lebesgue's dominated convergence theorem. Give consideration or 0.25 if the candidate notes that uniform integrability is useful in this context. Subtract 0.2 for mistakes.

Problem 4 Martingales (2 points)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = \sigma^2 < \infty$. Show that the sequence

$$M_n \triangleq \left(\sum_{i=1}^{n} X_i\right)^2 - \sigma^2 n, \quad n \geq 1,$$

is a martingale with respect to the filtration generated by $X_1, \ldots, X_n, n \geq 1$. Does $\left\{\frac{M_n}{n^2}\right\}_{n=1}^{\infty}$ converge in L^1 and almost surely? To what limit and why.

Solution: To show that $\{M_n\}_{n=1}^{\infty}$ is a martingale, we need to verify the three conditions:

1. $\{M_n\}_{n=1}^{\infty}$ is adapted.
2. M_n is integrable for all n.
3. the martingale property: for all $m \geq n \geq 1$ we have

$$\mathbb{E}[M_m|\mathcal{F}_n] = M_n.$$

Property one is trivial in this exercise since M_n is constructed as sums and powers of $X_1, \ldots, X_n$ and hence M_n is measurable w.r.t. $X_1, \ldots, X_n$ for each $n \geq 1$ and hence adapted.

Property two:

$$\mathbb{E}\left[\left|\left(\sum_{i=1}^{n} X_i\right)^2 - \sigma^2 n\right|\right] \leq C\sum_{i=1}^{n}\mathbb{E}\left[|X_i|^2\right] + C\sigma^2 n = 2Cn\sigma^2 < \infty$$

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for some constant C (independent of n but it is not so important).

Finally, the martingale property. It is enough to check for $m = n+1$ since we have $\mathcal{F}_n \subseteq \mathcal{F}_{m-1} \subseteq \mathcal{F}_m$ for $m \geq n+1$ and by the tower property we have

$$\mathbb{E}[M_m|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[M_m|\mathcal{F}_{m-1}]|\mathcal{F}_n] = \mathbb{E}[M_{m-1}|\mathcal{F}_n].$$

Now,

$$\begin{aligned}\mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n+1} X_i\right)^2 - \sigma^2(n+1)|\mathcal{F}_n\right] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i + X_{n+1}\right)^2 |\mathcal{F}_n\right] - \sigma^2(n+1).\end{aligned}$$

Now, $\sum_{i=1}^n X_i$ is $\mathcal{F}_n$-measurable and X_{n+1} is independent of $\mathcal{F}_n$. Hence, we can treat $\sum_{i=1}^n X_i$ as a constant and apply expectation on X_{n+1}. To do so and not get confused by what is the operator $\mathbb{E}$ applied on, we can write:

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i + X_{n+1}\right)^2 |\mathcal{F}_n\right] = \mathbb{E}\left[(y + X_{n+1})^2\right]\Big|_{y=\sum_{i=1}^n X_i}.$$

Now,

$$\mathbb{E}\left[(y + X_{n+1})^2\right] = y^2 + 2y\mathbb{E}[X_{n+1}] + \mathbb{E}[X_{n+1}^2] = y^2 + \sigma^2.$$

Back to the main equation, we arrive to

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i + X_{n+1}\right)^2 |\mathcal{F}_n\right] - \sigma^2(n+1) = \left(\sum_{i=1}^{n} X_i\right)^2 + \sigma^2 - \sigma^2(n+1) = M_n.$$

as we wanted to show.

We have,

$$\frac{M_n}{n^2} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 - \frac{\sigma^2}{n}.$$

Since by the strong law of large numbers $\frac{1}{n}\sum_{i=1}^n X_i \to \mathbb{E}[X_1]$ and $\mathbb{E}[X_1] = 0$ then we have

$$\frac{M_n}{n^2} \xrightarrow{a.s.} 0$$

as n goes to infinity.

Almost sure convergence does not imply L^1-convergence, but we may try to check L^1-convergence directly. Indeed,

$$\mathbb{E}\left[\left|\frac{M_n}{n^2}\right|\right] = \mathbb{E}\left[\left|\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 - \frac{\sigma^2}{n}\right|\right].$$

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Furthermore, we have

$$\mathbb{E}\left[\left|\frac{M_n}{n^2}\right|\right] \leq \mathbb{E}\left[\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i\right)^2\right] + \frac{\sigma^2}{n} = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}[X_i^2] + \frac{\sigma^2}{n},$$

since $\mathbb{E}[X_i X_j] = 0$, $i \neq j$, since they are independent with 0 mean. Hence,

$$\mathbb{E}\left[\left|\frac{M_n}{n^2}\right|\right] \leq \frac{\mathbb{E}[X_1^2]}{n} + \frac{\sigma^2}{n},$$

which goes to 0 as $n \to \infty$.

One can also prove the L^1-convergence here by using the second martingale convergence theorem on the submartingale M_n/n^2, since $\frac{1}{n}\sum_{i=1}^{n} X_i$ is uniformly integrable.

Points: Give 1 for showing the martingale part and 1p for the rest. Give 0.5 for a.s. convergence and 0.5 for L^1-convergence. Give 0.25 for mentioning strong law of large numbers. Give 0.25 for dominated convergence. Subtract 0.2 for mistakes.

GOOD LUCK!

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