UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK.MAT3710/4710 — Probability Theory
Day of examination:	16th December 2024
Examination hours:	3:00 pm – 7:00 pm
This problem set consists of 9 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature and the three remaining ones are more applied. All items within a problem are worth the same unless otherwise specified. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Grading: The total score is 10 points. The grading scale is F [0,4), E [4,5), D [5, 6), C [6, 7), B [7, 8.5), A [8.5,10].

Problem 1 Theory (2 points)

Let E be an uncountable set. Define the class of subsets of E given by

$$\mathcal{E} = \{A \subseteq E : \text{ either } A \text{ is countable, or } A^c \text{ countable}\}.$$

(a) Show that if $A, B \subseteq E$ are uncountable such that $A \cap B = \emptyset$, then either $A \notin \mathcal{E}$ or $B \notin \mathcal{E}$.

Solution: Assume that A and B are uncountable *and* both belong to $\mathcal{E}$ then it must be the case that A^c and B^c are countable. But then we would have $A^c \cup B^c = (A \cap B)^c = \emptyset^c = E$ and we would have that the union of two countable sets, i.e. A^c and B^c, gives an uncountable set, i.e. E, which is not possible. Hence A and B cannot both belong to $\mathcal{E}$. **Points:** Give 2/3p for this item. Subtract 0.2 for mistakes.

(b) Show that $\mathcal{E}$ is a σ-algebra.

Solution: We need to verify three conditions:

1. $\emptyset \in \mathcal{E}$ since the empty set is countable.
2. let $\{A_i\}_{i=1}^{\infty} \subset \mathcal{E}$ be disjoint, then if all the sets are countable we have $\cup_{i=1}^{\infty} A_i$ is countable and hence in $\mathcal{E}$. If one of them is uncountable, say A_j, then the union is uncountable. Note that

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we can only allow one of them being uncountable since they are disjoint, in virtue of item (a). However,

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

is countable since it is the intersection of sets, where A_j^c is countable. If the sets are not disjoint, we can select a family of disjoint sets such that their unions are equal.

3. Finally, $A \in \mathcal{E}$ and if A is countable then A^c is uncountable with countable complement and vice versa. Hence $A^c \in \mathcal{E}$.

Points: Give 2/3p for this item. Give 0.33p for a properly written definition of σ-algebra. Give 0.33 for verifying the definition. Subtract 0.2 for mistakes.

(c) Define the mapping $\mu : \mathcal{E} \to [0, \infty]$ by $\mu(X) = \mathbb{I}_{\{X^c \text{ countable}\}}$. Show that μ is a probability measure.

Solution: We need to verify three conditions:

1. $\mu() = 0$ since $\emptyset \in \mathcal{E}$ is countable.
2. let $\{A_i\}_{i=1}^{\infty} \subset \mathcal{E}$ be disjoint, then if all A_i's are countable we have that the union is countable and hence

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0.$$

But if one A_j is uncountable then the union is uncountable and hence

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 1.$$

On the other hand, if all A_i are countable then $\mu(A_i) = 0$ for all i and hence

$$\sum_{i=1}^{\infty} \mu(A_i) = 0.$$

But if only one, say A_j is uncountable then $\mu(A_i) = 0$ for all $i \neq j$ and $\mu(A_j) = 1$ and hence

$$\sum_{i=1}^{\infty} \mu(A_i) = 1.$$

As a result,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

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3. Finally, $\mu(E) = 1$ since E is uncountable.

Points: Give 2/3p for this item. Give 0.33p for a properly written definition of measure. Give 0.33 for verifying the definition. Subtract 0.2 for mistakes.

Problem 2 Random variables (3 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\{X_n\}_{n=1}^{\infty}$ a sequence of independent identically distributed random variables with uniform distribution on $[0,1]$, i.e. with densities $f_{X_n}(x) = 1$, $x \in [0,1]$, $n \geq 1$.

Define the random variables

$$L_{(1:n)} = \min\{X_1, \ldots, X_n\}, \quad H_{(1:n)} = \max\{X_1, \ldots, X_n\}, \quad n \geq 2.$$

It may not be surprising that $L_{(1:n)}$ and $H_{(1:n)}$ are not independent and, in fact, it can be shown that their joint density is given by

$$f_{(L_{(1:n)}, H_{(1:n)})}(l,h) = n(n-1)(h-l)^{n-2}, \quad 0 \leq l < h \leq 1.$$

(a) Prove that the density function of the range, denoted by R_n, of n independent uniform observations is given by

$$f_{R_n}(r) = n(n-1)r^{n-2}(1-r), \quad 0 < r < 1.$$

Solution: The range is $R_n = H_{(1:n)} - L_{(1:n)}$. Let $(u,v) = g(l,h) = (l, h-l)$ then $g^{-1}(u,v) = (u, u+v)$. Then $|Dg^{-1}(u,v)| = 1$ and $0 \leq l < h \leq 1$, $0 \leq u+v \leq 1$ and $0 \leq u, v \leq 1$. Then

$$f_{(U,V)}(u,v) = f_{(L,H)}(g^{-1}(u,v)) = n(n-1)v^{n-2}, \quad 0 \leq u+v \leq 1.$$

Now to get R we need to integrate away U, i.e.

$$f_V(v) = \int_0^1 n(n-1)v^{n-2}\mathbb{I}_{\{0 \leq u+v \leq 1\}}du = n(n-1)v^{n-2}(1-v), \quad 0 \leq v \leq 1,$$

which is exactly the density given in the exercise.

Points: Give 0.25p for a properly written formula of change of variables. Give 0.5 for finding the joint transformation but not the marginal. Subtract 0.2 for mistakes.

(b) Study the convergence of the sequence $\{R_n\}_{n=1}^{\infty}$ in distribution, probability, almost surely and in L^p, $p \geq 1$. Note: $\sum_{n=1}^{\infty} nr^n < \infty$ for every real r such that $|r| < 1$.

Solution: We have that

$$F_{R_n}(r) = \int_0^r f_{R_n}(u)du = nr^n\left[\frac{1}{r} - \frac{n-1}{n}\right],$$

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which tends to 0 as n tends to infinity as long as $r < 1$, when $r \geq 1$ we have that the limit is 1. Hence $R_n \to 1$ in distribution.

Moreover,

$$\mathbb{P}[|R_n - 1| > \varepsilon] = \mathbb{P}[R_n \leq 1 - \varepsilon] = n(1-\varepsilon)^n \left[\frac{1}{1-\varepsilon} - \frac{n-1}{n}\right]$$

which converges to zero by the previous argument since $1 - \varepsilon < 1$. Hence, R_n converges to 1 in probability, as well. Note that we knew this already, since 1 is a constant and convergence in distribution to a constant implies convergence in probability.

For a.s. convergence we aim to apply Borel-Cantelli lemma. Define $A_n = \{|R_n - 1| > \varepsilon\}$. We have already computes $\mathbb{P}[A_n]$. Namely,

$$\mathbb{P}[A_n] = nr^n \left[\frac{1}{1-\varepsilon} - \frac{n-1}{n}\right] \leq \frac{n(1-\varepsilon)^n}{1-\varepsilon}$$

and for every $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] \leq \sum_{n=1}^{\infty} \frac{n(1-\varepsilon)^n}{1-\varepsilon} < \infty,$$

and by Borel-Cantelli's first lemma we have

$$\mathbb{P}\left[\limsup_n \{|R_n - 1| > \varepsilon\}\right] = 0$$

and a.s. convergence follows.

Now, since $|R_n - 1|^p \leq C$ is bounded for every $p \geq 1$, we have by Lebesgue's dominated convergence theorem that

$$\lim_n \mathbb{E}[|R_n - 1|^p] = \mathbb{E}[\lim_n |R_n - 1|^p] = 0,$$

where we used continuity of the function $|\cdot|^p$ and almost sure convergence.

Points: Give 0.25p per convergence type (when justified). Give 0.25 for Borel-Cantelli (written theorem, not just name). Subtract 0.2 for mistakes.

(c) Consider independent observations $X_1, X_2, \ldots$ of the uniform distribution. What is the probability that we observe k consecutive observations that are, at most, within distance $\delta > 0$?

Solution: We can divide the observations in string of length k, i.e. $X_1, \ldots, X_k$ then $X_{k+1}, \ldots, X_{2k}$ and so on. Hence, we look at strings

$$\{X_{nk+1}, \ldots, X_{(n+1)k}\}_{n=1}^{\infty}.$$

Any of these string will consist of all observations being at most δ away from each other if their range is less than δ, i.e. $H_{(nk+1:(n+1)k)} -$

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$L_{(nk+1:(n+1)k)} < \delta$. Note that since the observations are i.i.d. this is the same as looking at the event $R_k < \delta$. We are asking whether the event $\{R_k < \delta\}$ for fixed k will occurs infinitely many times or not. But

$$\sum_{n=1}^{\infty} \mathbb{P}[R_k < \delta] = \infty$$

and hence by Borel-Cantelli's second lemma, since R_k are independent, we conclude that there will be infinitely many strings of length k that are at most a distance of δ from each other, for any arbitrary k.

Points: Give 0.25p if there is an attempt to apply Borel-Cantelli. Give 0.25 for Borel-Cantelli (written theorem, not just name). Subtract 0.2 for mistakes.

Problem 3 Conditional expectation (2 points)

Let X and Y be two independent Bernoulli distributed random variables in some probability space, i.e. $X, Y \in \{0, 1\}$ and $\mathbb{P}[X = 1] = \mathbb{P}[Y = 1] = p$ for some $p \in [0, 1]$. Define the random variable $Z = \mathbb{I}_{\{X+Y=0\}}$.

(a) Compute $\mathbb{E}[Z]$, $\mathbb{E}[X|Z]$ and verify that $\mathbb{E}[\mathbb{E}[X|Z]] = \mathbb{E}[X]$.

Solution:

$\mathbb{E}[Z] = \sum_{x,y\in\{0,1\}} \mathbb{I}_{\{X+Y=0\}} \mathbb{P}[X = x]\mathbb{P}[Y = y] = \mathbb{P}[X = 0]\mathbb{P}[Y = 0] = (1 - p)^2$.

To compute $\mathbb{E}[X|Z]$ note that the σ-algebra generated by Z is finite, i.e. $\{Z = 0\}$ and $\{Z = 1\}$ then let us compute the conditional expectation on these two events:Note that Z is also Bernoully with probability $\mathbb{P}[Z = 1] = (1 - p)^2$. Hence,

$$\begin{aligned}
\mathbb{E}[X|Z = 0] &= \frac{\mathbb{E}[X\mathbb{I}_{\{Z=0\}}]}{\mathbb{P}[Z = 0]} \\
&= \frac{\mathbb{P}[X = 1]\mathbb{P}[Y = 1] + \mathbb{P}[X = 1]\mathbb{P}[Y = 0]}{1 - (1 - p)^2} \\
&= \frac{p^2 + p(1 - p)}{1 - (1 - p)^2} \\
&= \frac{1}{2 - p}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}[X|Z = 1] &= \frac{\mathbb{E}[X\mathbb{I}_{\{Z=1\}}]}{\mathbb{P}[Z = 1]} \\
&= \frac{X\mathbb{I}_{\{X+Y=0\}}}{(1 - p)^2} = 0.
\end{aligned}$$

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In a summary,

$$\mathbb{E}[X|Z=z] = \frac{1}{2-p}\mathbb{I}_{\{z=0\}}$$

and hence

$$\mathbb{E}[X|Z] = \frac{1-Z}{2-p}.$$

Now, we have by linearity of expectation,

$$\mathbb{E}[\mathbb{E}[X|Z]] = \frac{1-\mathbb{E}[Z]}{2-p} = \frac{1-(1-p)^2}{2-p} = p$$

which also coincides with $\mathbb{E}[X] = p$.

Points: Give 0.5p for $\mathbb{E}[X|Z]$ and 0.25p for the two other. Give 0.25p if someone writes down the definition of conditional expectation perfectly. Subtract 0.2 for mistakes. In particular, subtract 0.2 if candidate does not express $\mathbb{E}[X|Z]$ as a Borel-measurable function of Z.

(b) Compute $\mathbb{E}[Z|X]$. Is it true that $\mathbb{E}[\mathbb{E}[X|Z]|X] = \mathbb{E}[\mathbb{E}[Z|X]|Z]$? What about $\mathbb{E}[\mathbb{E}[X|Z]|X,Z] = \mathbb{E}[X|Z]$? Justify your answer.

Solution: We have,

$$\begin{aligned}\mathbb{E}[Z|X=x] &= \sum_{z\in\{0,1\}} z\mathbb{P}[Z=z|X=x] \\ &= \mathbb{P}[Z=1|X=x] \\ &= \mathbb{P}[X+Y=0|X=x] \\ &= \mathbb{P}[x+Y=0] \\ &= (1-p)(1-X).\end{aligned}$$

hence,

$$\mathbb{E}[Z|X] = (1-p)(1-X).$$

Now, we have on the one hand,

$$\mathbb{E}[\mathbb{E}[X|Z]|X] = \mathbb{E}\left[\frac{1-Z}{2-p}\middle|X\right] = \frac{1-\mathbb{E}[Z|X]}{2-p} = \frac{1-(1-p)(1-X)}{2-p}$$

and on the other hand,

$$\mathbb{E}[\mathbb{E}[Z|X]|Z] = \mathbb{E}[(1-p)(1-X)|Z] = (1-p)(1-\mathbb{E}[X|Z]) = (1-p)\left(1-\frac{1-Z}{2-p}\right).$$

These random quantities are always different for all $p \in [0,1]$.

It is true that $\mathbb{E}[\mathbb{E}[X|Z]|X,Z] = \mathbb{E}[X|Z]$ since $\sigma(Z) \subseteq \sigma(X,Z)$ and by the stability property we get the result.

Points: Give 0.5p for $\mathbb{E}[Z|X]$ and 0.25p for the questions. Give 0.25p if someone writes down the definition of conditional expectation perfectly, but only once (not both in (a) and (b)). Subtract 0.2 for mistakes. In particular, subtract 0.2 if candidate does not express conditional expectations as Borel-measurable functions.

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Problem 4 Martingales (3 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and X, Y two independent random variables in it. Assume that X is uniformly distributed on $[0,1]$, i.e. $f(x) = 1$ for $x \in [0,1]$ and $\mathbb{P}[Y = -1] = \mathbb{P}[Y = 1] = \frac{1}{2}$. Consider the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(X)$ and $\mathcal{F}_n = \sigma(X, Y)$ for al $n \geq 2$. Define for each $n \geq 1$ the processes

$$M_n = \frac{Y}{X}\mathbb{I}_{\{n \geq 2\}}$$

and

$$\tau_n = \begin{cases} 1, & n < \frac{1}{X}, \\ \infty, & n \geq \frac{1}{X} \end{cases} .$$

(a) Show that $\{M_n\}_{n \geq 1}$ is not a martingale.

Solution: A martingale needs to satisfy three conditions: adaptedness, integrability and the martingale property. In this case, we see that M_n is adapted to the given filtration. However, the second property breaks down for $n \geq 2$. Indeed, take $n \geq 2$,

$$\mathbb{E}[|M_n|] = \mathbb{E}\left[\frac{|Y|}{|X|}\right] = \mathbb{E}[1/X] = \infty,$$

since $|Y| = 1$ and the inverse of the uniform has infinite expectation.

Points: Give 1p for this item. Give 0.5p if candidate writes down definition of martingale perfectly. Subtract 0.2 for mistakes.

(b) Show that $\{\tau_n\}_{n \geq 1}$ is a monotone sequence of stopping times which converges almost surely to infinity. Note: we have defined stopping times as being finite. Here, we allow infinity to be an outcome. The definition remains the same with the additional requirement that $\{\tau_n = \infty\}$ also needs to be measurable.

Solution: For each $n \geq 1$ we have that for every $k \in \mathbb{N}$,

$$\{\tau_n \leq k\} = \{\tau_n = 1\} = \left\{X < \frac{1}{n}\right\} \in \mathcal{F}_n$$

for every n. Also,

$$\{\tau_n = \infty\} = \{X \geq 1/n\} \in \mathcal{F}_n$$

hence τ_n is a stopping time for every $n \geq 1$.

It is monotone non-decreasing,

$$\begin{aligned}\mathbb{P}[\tau_n \leq \tau_{n+1}] &= \mathbb{P}[\tau_n = 1, \tau_{n+1} = 1] + \mathbb{P}[\tau_n = 1, \tau_{n+1} = \infty] + \mathbb{P}[\tau_n = \infty, \tau_{n+1} = \infty] \\ &= \mathbb{P}\left[X < \frac{1}{n}, X < \frac{1}{n+1}\right] + \mathbb{P}\left[X < \frac{1}{n}, X \geq \frac{1}{n+1}\right] + \mathbb{P}\left[X \geq \frac{1}{n}, X \geq \frac{1}{n+1}\right] \\ &= \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+1} + 1 - \frac{1}{n} = 1.\end{aligned}$$

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Finally, $\frac{1}{\tau_n} = \mathbb{I}_{\{n<\frac{1}{X}\}}$ which clearly converges to 0 almost surely (actually surely). Hence $\tau_n \to \infty$ almost surely (actually surely).

Points: Give 1p for this item. Give 0.5 for stopping time and 0.25 for monotonicity and 0.25 for limit Give 0.25p if candidate writes down definition of stopping time perfectly. Subtract 0.2 for mistakes.

(c) Prove that $\{M_{\tau_k \wedge n}\}_{n\geq 1}$ is a martingale for every $k \geq 1$. Here, $x \wedge y \triangleq \min\{x, y\}$. A process like $\{M_n\}_{n\geq 1}$ is known as local martingale.

Solution: We need to check the three properties for martingale, but first let us find a simpler expression for $M_{\tau_k \wedge n}$. Since τ_k is either 1 of ∞ then $\tau_k \wedge n$ is either 1 or n for $n \geq 1$. We have $M_{\tau_k \wedge 0} = M_0 = 0$ and $M_{\tau_k \wedge 1} = M_1 = 0$ as well. Now, for every $n \geq 2$,

$$M_{\tau_k \wedge n} = \begin{cases} M_1 & \text{if} \quad \tau_k = 1, \\ M_n & \text{if} \quad \tau_k = \infty. \end{cases} = \begin{cases} 0 & \text{if} \quad X < \frac{1}{k}, \\ M_n & \text{if} \quad X \geq \frac{1}{k}. \end{cases},$$

since $M_1 = 0$.

In conclusion,

$$M_{\tau_k \wedge n} = M_n \mathbb{I}_{\{X \geq \frac{1}{k}\}} = \frac{Y}{X} \mathbb{I}_{\{X \geq \frac{1}{k}\}},$$

for $n \geq 2$ which is clearly adapted since $\frac{Y}{X}$ and $\mathbb{I}_{\{X \geq \frac{1}{k}\}}$ are $\mathcal{F}_n = \sigma(X, Y)$-measurable for $n \geq 2$, i.e. $M_{\tau_k \wedge n}$ is $\mathcal{F}_n$-measurable for each $n \geq 1$.

Moreover, for $n = 1$, $M_{\tau_k \wedge n} = 0$ and hence integrable and for $n \geq 2$,

$$\mathbb{E}[|M_{\tau_k \wedge n}|] = \mathbb{E}\left[\frac{|Y|}{X} \mathbb{I}_{\{X \geq \frac{1}{k}\}}\right] = \mathbb{E}\left[\frac{1}{X} \mathbb{I}_{\{X \geq \frac{1}{k}\}}\right] = \int_{1/k}^{1} \frac{1}{x} dx = \log k < \infty$$

for every $k \geq 1$ and $n \geq 2$.

Finally, the martingale property. We have $M_{\tau_k \wedge 1} = M_1 = 0$ so we start for $n \geq 1$,

$$\mathbb{E}[M_{\tau_k \wedge (n+1)} | \mathcal{F}_n] = \mathbb{E}[M_{n+1} \mathbb{I}_{\{X \geq \frac{1}{k}\}} | \mathcal{F}_n].$$

For $n = 1$ we have

$$\mathbb{E}[M_2 \mathbb{I}_{\{X \geq \frac{1}{k}\}} | \sigma(X)] = \mathbb{E}\left[\frac{Y}{X} \mathbb{I}_{\{X \geq \frac{1}{k}\}} | \sigma(X)\right].$$

Since $\frac{1}{X} \mathbb{I}_{\{X \geq \frac{1}{k}\}}$ is $\sigma(X)$-measurable and Y is independent of X we have

$$\mathbb{E}[M_2 \mathbb{I}_{\{X \geq \frac{1}{k}\}} | \sigma(X)] = \frac{1}{X} \mathbb{I}_{\{X \geq \frac{1}{k}\}} \mathbb{E}[Y] = 0 = M_{\tau_k \wedge 1}.$$

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Now, for $n \geq 2$,

$$\mathbb{E}[M_{\tau_k \wedge (n+1)} | \mathcal{F}_n] = \mathbb{E}\left[\frac{Y}{X} \mathbb{I}_{\{X \geq \frac{1}{k}\}} | \sigma(X,Y)\right] = \underbrace{\frac{Y}{X}}_{=M_n} \mathbb{I}_{\{X \geq \frac{1}{k}\}} = M_{\tau_k \wedge n}$$

and hence $M_{\tau_k \wedge n}$, $n \geq 1$ is a martingale w.r.t. the given filtration.

Points: Give 1p for this item. Give 0.5p if candidate writes down definition of martingale perfectly (if not done in item (a)). Give 0.25 points for adaptedness, 0.25 for integrability and 0.5 for martingale property. Subtract 0.2 for mistakes.

GOOD LUCK!