Fordelingen til $\bar{X}$ når X_i - ene er $N(\mu, \sigma^2)$ vanh.

Har $M_{X_i}(t) = E(e^{tX_i}) = \exp(\mu t + \sigma^2 t^2/2)$

Finner

$$M_{\bar{X}}(t) = E\left(e^{t\bar{X}}\right) = E\left(\exp\left(t\left(\tfrac{1}{n}X_1 + \tfrac{1}{n}X_2 + \cdots + \tfrac{1}{n}X_n\right)\right)\right)$$

$$= E\left[e^{\frac{t}{n}X_1} \cdot e^{\frac{t}{n}X_2} \cdot \ldots \cdot e^{\frac{t}{n}X_n}\right]$$

$$= E\left(e^{\frac{t}{n}X_1}\right) \cdot E\left(e^{\frac{t}{n}X_2}\right) \cdot \ldots \cdot E\left(e^{\frac{t}{n}X_n}\right)$$

$$= \exp\left(\mu\tfrac{t}{n} + \tfrac{\sigma^2 t^2}{n^2 \cdot 2}\right) \cdot \ldots \cdot \exp\left(\mu\tfrac{t}{n} + \tfrac{\sigma^2 t^2}{n^2 \cdot 2}\right)$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{n 2}\right)$$

Dette er MGF til $N\left(\mu, \frac{\sigma^2}{n}\right)$

Derfor er $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

Chebycheus ulikhet

Y stok. var. $\quad E(Y) = \mu \quad V(Y) = \sigma^2$

$$\sigma^2 = \int_{-\infty}^{\infty} (y-\mu)^2 f(y)\,dy \qquad \text{(pr. def.)}$$

midtre side omrade

$$\geq \int_{y:\,|y-\mu| \geq k\sigma} (y-\mu)^2 f(y)\,dy$$

ende midtre

$$\geq \int_{y:\,|y-\mu| \geq k\sigma} (k\sigma)^2 f(y)\,dy$$

$$= k^2\sigma^2 \int_{y:\,|y-\mu| \geq k\sigma} f(y)\,dy = k^2\sigma^2 \, P(|Y-\mu| \geq k\sigma)$$

Dermed

$$P(|Y-\mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}$$

Gjennomsnitt når n øker

$X_1, X_2, \ldots X_n$ uavh. $\quad E(X_i) = \mu \quad V(X_i) = \sigma^2$

$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad E(\bar{X}_n) = \mu \quad V(\bar{X}_n) = \frac{\sigma^2}{n}$

$SD(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}$

Chebychev gir:

$$P(|\bar{X}_n - \mu| \geq k \cdot SD(\bar{X}_n)) \leq \frac{1}{k^2}$$

La $\varepsilon > 0$. Sett $k \cdot SD(\bar{X}_n) = \varepsilon$

dvs. $\boxed{k = \frac{\varepsilon}{SD(\bar{X}_n)}}$

Får da

$$P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{1}{k^2} = \frac{1}{(\varepsilon / SD(\bar{X}_n))^2} = \frac{1}{(\varepsilon / \sigma / \sqrt{n})^2}$$

$$= \frac{\sigma^2}{\varepsilon^2 n}$$

Dermed

$$P(|\bar{X}_n - \mu| < \varepsilon) = 1 - P(|\bar{X}_n - \mu| \geq \varepsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2 n}$$

Dermed

$$\boxed{\lim_{n \to \infty} P(|\bar{X}_n - \mu| < \varepsilon) = 1}$$

↑

Store tall lov!

Bare vi har "mange nok" observasjoner, vil gjennomsnittet av dem nærme seg forventningsverdien.

Kap. 5.3 : formelsaml. "Addisjonsregel"

Fordeling til lineærkombinasjoner og summer

$X_1 \dots X_n$ uavh. $\quad M_{X_i}(t) = E(e^{tX_i})$

Ser på $\quad Y = a_1X_1 + \dots + a_nX_n$

MGF til Y

$$M_Y(t) = E(e^{tY}) = E(e^{t(a_1X_1 + \dots + a_nX_n)})$$

$$= E(e^{a_1tX_1} \cdot \dots \cdot e^{a_ntX_n})$$

$$= E(e^{a_1tX_1}) \cdot E(e^{a_2tX_2}) \cdot \dots \cdot E(e^{a_ntX_n})$$

$$= M_{X_1}(a_1t) \cdot \dots \cdot M_{X_n}(a_nt) = \prod_{i=1}^{n} M_{X_i}(a_it)$$

↑ multiplikasjons-symbol

Normalfordeling

$X_i \sim N(\mu_i, \sigma_i^2)$

$M_{X_i}(t) = \exp(\mu_i t + \sigma_i^2 t^2/2)$

Dermed for $Y = a_1X_1 + \dots + a_nX_n$

$$M_Y(t) = \prod_{i=1}^{n} \exp(\mu_i a_i t + \sigma_i^2 (a_i t)^2/2)$$

$$= \exp\left((\Sigma a_i\mu_i)t + (\Sigma a_i^2\sigma_i^2)t^2/2\right)$$

Derfor er

$$Y = a_1X_1 + a_2X_2 + \dots + a_nX_n \sim N(\Sigma a_i\mu_i, \Sigma a_i^2\sigma_i^2)$$

Når $a_i = \frac{1}{n}$ og $\mu_i = \mu$ og $\sigma_i^2 = \sigma^2$

får vi $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ som vist i første time.

Poissonfordeling

$X_1, \ldots X_n$ uavh. $X_i \sim \text{Poisson}(\lambda_i)$

Har $M_{X_i}(t) = \exp(\lambda_i(e^t - 1))$

Ser på $Y = X_1 + X_2 + \ldots + X_n$ (bare sum, div. $a_i = 1 \ \forall i$)

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \exp(\lambda_i(e^t - 1)) = \exp\left(\left(\sum \lambda_i\right)(e^t - 1)\right)$$

Så $Y = \sum_{i=1}^{n} X_i \sim \text{Poisson}\left(\sum_{i=1}^{n} \lambda_i\right)$