Nøkkelbegreper kap. 7.1 - 7.2

Estimering

$X_1, \ldots X_n$ u.i.f. $\sim f(\cdot\,; \theta)$

θ kan være en vektor

Estimator - estimat

$\hat{\theta} = (X_1, \ldots X_n)$ $\hat{\theta}(x_1, \ldots x_n)$

MSE - bruttovarians

$$MSE(\hat{\theta}) = E\left[(\hat{\theta} - \theta)^2\right]$$

$$= V(\hat{\theta}) + \text{skjevhet}^2$$

$$\text{skjevhet}^2 = \left(E(\hat{\theta}) - \theta\right)^2$$

Forventningsrett estimator:

$E(\hat{\theta}) = \theta$, dvs. ingen skjevhet

dvs. ingen bias

$X_1 \ldots X_n$ u.i.f. $E(X_i) = \mu$ $V(X_i) = \sigma^2$

Da er

$$\hat{\mu} = \bar{X} \quad \text{og} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

forventningsrette estimatorer for hhv. μ og σ^2

— konsistente ulike

Konsistens

Standardfeil - parametrisk bootstrap

Kap. 7.2 Metoder for å utlede/finne estimatorer

- Momentmetoden
- Maximum Likelihoodmetoden

Parametisk bootstrap

for å finne standardfeilen til en estimator $\hat{\theta}$

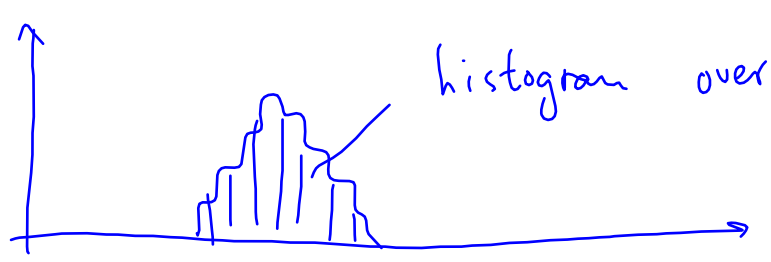
$X_1, \ldots X_n$ u.i.f. $\sim f(\cdot\,; \theta)$

Estimerer θ med estimator $\hat{\theta}$. Ønsker å estimere standardfeilen til $\hat{\theta}$, dvs. $\sigma_{\hat{\theta}}$

Param. bootstrap skissert:

1) Trekk B utvalg av størrelse n fra $f(\cdot\,; \hat{\theta})$

2) Estimer B ulike $\hat{\theta}_b$ $(b = 1, \ldots B)$
fra hvert av de simulerte utvalgene



3) Regn ut

$$s_{\hat{\theta}} = \hat{\sigma}_{\hat{\theta}} = \sqrt{\frac{1}{B-1} \sum_{b=1}^{B} (\hat{\theta}_b - \bar{\theta}_b)^2}$$

der $\bar{\theta}_b = \frac{1}{B} \sum_{b=1}^{B} \hat{\theta}_b$

Dette gir et estimat for standardfeilen til $\hat{\theta}$

Estimering i kont. uniform fordeling

$X_1 \dots X_n$ u.i.f. $\sim$ Unif $[0, \theta]$

$$f(x) = \begin{cases} 1/\theta & 0 \le x \le \theta \\ 0 & \text{ellers} \end{cases}$$

Har at $E(X_i) = \theta/2$ og $V(X_i) = \theta^2/12$ (vist tidligere i kurset)

En mulig estimator for θ er

$\theta^* = 2\bar{X}$ $\qquad E(\bar{X}) = \frac{\theta}{2}$ ←

$V(\bar{X}) = \frac{\theta^2}{12 \cdot n}$ ←

Har da at

$$E(\theta^*) = E(2\bar{X}) = 2E(\bar{X}) = 2 \cdot \frac{\theta}{2} = \theta \quad \text{forv. rett!}$$

$$V(\theta^*) = V(2\bar{X}) = 4V(\bar{X}) = \frac{4 \cdot \theta^2}{12 \cdot n} = \frac{\theta^2}{3n}$$

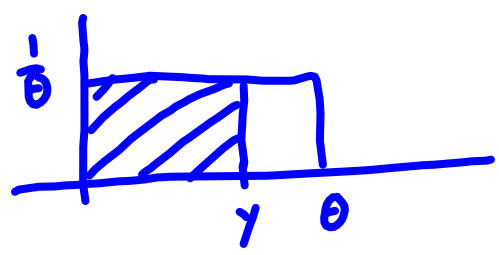
Alternativ estimator: tar utgangspunkt i $Y_n = \max X_i$

Kumulativ fordeling for Y_n:

$$F_{Y_n}(y) = P(Y_n \le y) = P(\max X_i \le y)$$

husk!: $= P(\text{alle} \le y) = P(X_1 \le y) \cdot P(X_2 \le y) \cdot \dots \cdot P(X_n \le y)$

$$= \frac{y}{\theta} \cdot \frac{y}{\theta} \cdot \dots \cdot \frac{y}{\theta} = \left(\frac{y}{\theta}\right)^n$$



Tetthetene til Y_n blir da

$$f_{Y_n}(y) = F'_{Y_n}(y) = \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1} \qquad 0 \le y \le \theta$$

Finner

$$E(Y_n) = \int_0^\theta y f_{Y_n}(y)\,dy = \int y \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1} dy = \frac{n}{n+1}\theta$$

$$E(Y_n^2) = \int_0^\theta y^2 f_{Y_n}(y)\,dy = \cdots = \frac{n}{n+2}\theta^2$$

$$V(Y_n) = E(Y_n^2) - \left[E(Y_n)\right]^2$$

$$= \frac{n\theta^2}{(n+1)^2(n+2)}$$

Setter $\hat{\theta} = \frac{n+1}{n} Y_n$ gir

$$\underline{\underline{E(\hat{\theta})}} = E\left[\frac{n+1}{n} Y_n\right] = \frac{n+1}{n} E(Y_n)$$

$$= \frac{n+1}{n}\,\frac{n}{n+1}\,\theta = \theta$$ så

$\hat{\theta}$ er forv. rett!

$$V(\hat{\theta}) = \left(\frac{n+1}{n}\right)^2 V(Y_n) = \cdots = \underline{\underline{\frac{\theta^2}{n(n+2)}}}$$

Momentmetoden

1) Bernoulli (p)

$X_1 \dots X_n$ u.i.f. Bernoulli (p)

$E(X_i) = p$

$P(X_i = 1) = p$

$P(X_i = 0) = 1 - p$

$m = 1$ (bare én parameter)

Teoretisk første moment $E(X_i) = p$

Empirisk — " — $\frac{1}{n}\sum_{i=1}^{n} X_i$

$\hat{p} = \frac{1}{n}\sum X_i$ er en momentestimator for p

2) Gamma (α, β)

$X_1 \dots X_n$ u.i.f. gamma (α, β)

$m = 2$ (to parametre)

Vi vet teoretiske momenter:

$E(X_i) = \alpha\beta$

$E(X_i^2) = V(X_i) + [E(X_i)]^2$

$= \alpha\beta^2 + \alpha^2\beta^2$

$= \alpha\beta^2(1+\alpha)$

to ligninger med to ukjente

$\overline{X}$

$\frac{1}{n}\sum X_i = \alpha\beta$

$\frac{1}{n}\sum X_i^2 = \alpha\beta^2(1+\alpha)$

$\overline{X^2}$

$\overline{X^2} = \frac{1}{n}\sum X_i^2$

$(\overline{X})^2 = \left(\frac{1}{n}\sum X_i\right)^2$

$\alpha = \frac{\overline{X}}{\beta}$

$\overline{X^2} = \frac{\overline{X}}{\beta}\beta^2\left(1 + \frac{\overline{X}}{\beta}\right)$

$\overline{X^2} = \overline{X}\beta + (\overline{X})^2$

$\hat{\beta} = \frac{\overline{X^2}}{\overline{X}} - \overline{X}$

Da blir

$\hat{\alpha} = \frac{\overline{X}}{\hat{\beta}} = \frac{(\overline{X})^2}{\overline{X^2} - (\overline{X})^2}$