Oppgave 1, Kapittel 2

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & & \vdots \\ \vdots & & p_{33} \ddots & \vdots \\ \vdots & & & \\ p_{N1} & \cdots & & p_{NN} \end{bmatrix}$$

Anta at P er en $N \times N$ stokastisk matrise,

dvs $\sum_{i=1}^{N} p_{ij} = 1 \quad \forall j$

a) Vis at P^2 er en stokastisk matrise:

$$P^2 = P \cdot P = \begin{bmatrix} p_{11} & \cdots & p_{1N} \\ p_{21} & & \vdots \\ \vdots & & \vdots \\ p_{N1} & & p_{NN} \end{bmatrix} \begin{bmatrix} p_{11} & \cdots & p_{1N} \\ p_{21} & & \\ & & \\ p_{N1} & & p_{NN} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{N} p_{1k} p_{k1} & \sum_{k=1}^{N} p_{1k} p_{k2} & \cdots & \sum_{k=1}^{N} p_{1k} p_{kN} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \sum_{k=1}^{N} p_{Nk} p_{k1} & & & \sum_{k=1}^{N} p_{Nk} p_{kN} \end{bmatrix}$$

dvs element (i,j) er $\sum_{k=1}^{N} p_{ik} p_{kj}$

$$\sum_{i=1}^{N} p_{ij}^{(2)} = \sum_{i=1}^{N}\sum_{k=1}^{N} p_{ik}\,p_{kj} = \sum_{k=1}^{N} p_{kj} \underbrace{\sum_{i=1}^{N} p_{ik}}_{=1 \text{ fordi } P \text{ er stokastisk}}$$

$$= \sum_{k=1}^{N} p_{kj} = 1 \quad \text{fordi } P \text{ er stokastisk}$$

Q.E.D.

Skal nå vise at P^n er stokastisk for alle $n > 0$.
Har allerede vist for $n=1$ & $n=2$.

Antar at P^{n-1} er stokastisk for $n-1 > 2$

dvs $\sum_{i=1}^{N} p_{ij}^{(n-1)} = 1 \quad \forall j$

$$P^n = P^{n-1} \cdot P = \begin{bmatrix} \sum_{k=1}^{N} p_{1k}^{(n-1)} p_{k1} & \cdots & \sum_{k=1}^{N} p_{1k}^{(n-1)} p_{kN} \\ \vdots & & \vdots \\ \sum_{k=1}^{N} p_{Nk}^{(n-1)} p_{k1} & \cdots & \sum_{k=1}^{N} p_{Nk}^{(n-1)} p_{kN} \end{bmatrix}$$

dvs element (i,j) er $\sum_{k=1}^{N} p_{ik}^{(n-1)} p_{kj}$

$$\sum_{i=1}^{N} \sum_{k=1}^{N} p_{ik}^{(n-1)} p_{kj} = \sum_{k=1}^{N} p_{kj} \underbrace{\sum_{i=1}^{N} p_{ik}^{(n-1)}}_{=1 \text{ fordi } P^{(n-1)} \text{ er stokastisk}} = \sum_{k=1}^{N} p_{kj} = 1$$

Q.E.D.

b.) Helt tilsvarende som a.), bare at her skal også alle radene summere seg til 1.

Fordi P er dobbelt stokastisk, dvs

$\sum_{j=1}^{N} p_{ij} = 1 \quad \forall i$, følger det at

$\sum_{j=1}^{N} p_{ij}^{(n)} = 1 \quad \forall_i$ helt tilsvarende som i a.)