Oppgave 5, Kap. 3

a.) Hvis man er i tilstand N er sannsynligheten for å bli absorbert i tilstand N 1, altså $b_N = 1$

$$b_0 = (1-p)\,b_0 + p b_1 = b_0 - p b_0 + p b_1$$

$$\Rightarrow \quad b_0 = b_1$$

b.) $b_k = p b_{k+1} + q b_{k-1} \quad , \quad 0 < k < N$

Differenseligning:

$$p b_{k+1} - b_k + q\,b_{k-1} = 0 \quad , \quad 0 < k < N$$

med initialbetingelser: $b_N = 1 \;,\; b_0 = b_1$

Skal vise at $b_k = 1$. Setter inn i ligningen for $0 < k < N$

$$p \cdot 1 - 1 + q \cdot 1 = p + q - 1 = 0 \quad \text{Stemmer!}$$

$$b_N = 1 \;,\; b_0 = b_1 = 1$$

c.) Ligning som bestemmer τ_k:

$$\tau_k = p\cdot(1+\tau_{k+1}) + q(1+\tau_{k-1}), \quad 0<k<N$$

$$= 1 + p\tau_{k+1} + q\tau_{k-1}$$

Initialbet. $\tau_N = 0$, $\quad \tau_0 = p(\tau_1+1) + q(\tau_0+1)$

$$= 1 + p\tau_1 + q\tau_0 = 1 + p\tau_1 + (1-p)\tau_0$$

$$\Rightarrow \frac{1}{p} + \tau_1 - \tau_0 = 0 \Rightarrow \tau_0 = \tau_1 + \frac{1}{p}$$

Setter inn, $p = q = \frac{1}{2}$:

$0 < k < N$:

$$N(N+1) - k(k+1) = 1 + \frac{1}{2}\left(N(N+1) - (k+1)(k+2)\right)$$

$$+ \frac{1}{2}\left(N(N+1) - (k-1)k\right)$$

$$= 1 + N(N+1) - \frac{1}{2}\left(k^2 + 2k + \cancel{k} + 2 + k^2 - \cancel{k}\right)$$

$$= 1 + N(N+1) - (k^2 + k + 1)$$

$$= N(N+1) - k(k+1) \quad \text{OK}$$

$k=0$:

$$N(N+1) - 0 = N(N+1) - 2 + 2 \quad \text{OK}$$

$k=N$: $\tau_N = 0$ OK

$p \neq q$: $\quad 0 < k < N$:

$$\underbrace{\frac{q}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k\right] - \frac{N-k}{q-p}}_{VS} = \underbrace{1 + \frac{pq}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^{k+1}\right]}_{HS}$$

$$- \frac{p(N-(k+1))}{q-p} + \frac{q^2}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^{k-1}\right]$$

$$- \frac{q(N-(k-1))}{q-p}$$

$$HS = 1 + \frac{q}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N \overbrace{(p+q)}^{=1}\right.$$

$$\left. - p\left(\frac{q}{p}\right)^{k+1} - q\left(\frac{q}{p}\right)^{k-1}\right] - \frac{p(N-(k+1)) + q(N-(k-1))}{q-p}$$

$$= 1 + \frac{q}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N - \frac{q^{k+1}}{p^k} - \frac{q^k}{p^{k-1}}\right] - \frac{N-k \pm (q-p)}{q-p}$$

$$= \frac{q}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N - \frac{q^{k+1} + \overset{(1-q)}{p} \cdot q^k}{p^k}\right] - \frac{N-k}{q-p}$$

$$= \frac{q}{(q-p)^2}\left(\left(\frac{q}{p}\right)^N - \frac{q^k(q+1-q)}{p^k}\right) - \frac{N-k}{q-p}$$

$$= \frac{q}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k\right] - \frac{N-k}{q-p} = VS$$

$k=0$:

$$\frac{q}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N - 1\right] - \frac{N}{q-p}$$

$$= \frac{q}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N - \frac{q}{p}\right] - \frac{N-1}{q-p} + \frac{1}{p}$$

$$= \frac{q}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N - 1\right] + \frac{q^2-p}{p(q-p)^2} - \frac{N-1}{q-p} + \frac{1}{p}$$

$$= \frac{q}{(q-p)^2}\left[\left(\frac{q}{p}\right)^N - 1\right) + \underbrace{\frac{-(q^2-p)+p(q-p)+(q-p)^2}{p(q-p)^2}}_{(*)} - \frac{N}{q-p}$$

$(*) = 0$, da er VS = HS

$(*)$ - teller:

$$-q^2 + p + pq - p^2 + q^2 - 2pq + p^2$$

$$= p - 2pq = p - 2p(1-p)$$

$$= p - 2p + p = 0 \quad \text{ok}$$

$k = N$: $\tau_N = 0$ ok !

Exercise 2, Chapter 3 — Ikke grenseovergang

Semi-infinite random walk, dvs tilstander $\{0,1,2,\ldots\}$

Finn a_k og τ_k rett fra random walk modell med absorberende boundaries ved å la $N \to \infty$ i (3.4), (3.5), (3.9) og (3.10)

(3.4): $$a_k = \frac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^N - 1}, \quad p \neq q$$

(3.5): $$a_k = \frac{N-k}{N}, \quad p = \frac{1}{2} = q$$

(3.9): $$\tau_k = \frac{1}{q-p}\left[k - N\left(\frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N}\right)\right], \quad p \neq q$$

(3.10): $$\tau_k = k(N-k), \quad p = \frac{1}{2} = q$$

Lar $\tilde{a}_k$ og $\tilde{\tau}_k$ være for semi-infinite random walk:

$$p \neq q: \quad \tilde{a}_k = \lim_{N\to\infty} a_k = \lim_{N\to\infty} \frac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^N - 1}$$

$$q < p: \quad \tilde{a}_k = \frac{0 - \left(\frac{q}{p}\right)^k}{0 - 1} = \underline{\underline{\left(\frac{q}{p}\right)^k}}$$

$$p > q: \quad \tilde{a}_k = \lim_{N\to\infty} \frac{q^N - \left(\frac{q}{p}\right)^N p^N}{q^N - p^N} = \frac{0}{0} = \underline{\underline{1}}$$

$$p = q = \frac{1}{2}: \quad \tilde{a}_k = \lim_{N\to\infty} \frac{N-k}{N} = \lim_{N\to\infty} \frac{1-\frac{k}{N}}{1} = \underline{\underline{1}}$$

$$p \neq q: \quad \tilde{\tau}_k = \lim_{N \to \infty} \frac{1}{q-p} \left[k - \frac{N\left(1-\left(\frac{q}{p}\right)^k\right)}{\left(1-\left(\frac{q}{p}\right)^N\right)} \right]$$

$$= \lim_{N \to \infty} \frac{1}{q-p} \left[k - \frac{\left(1-\left(\frac{q}{p}\right)^k\right)}{\frac{1}{N}\left(1-\left(\frac{q}{p}\right)^N\right)} \right]$$

$$q < p: \quad \tilde{\tau}_k = \infty$$

$$q > p: \quad \tilde{\tau}_k = \frac{k}{q-p}$$

$$p = q = \frac{1}{2}: \quad \tilde{\tau}_k = \lim_{N \to \infty} k(N-k) = \underline{\underline{\infty}}$$