Exercise 3, Chapter 5

$$Q = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$$

a.) Vis at $Q^n = (-3)^{n-1} Q$

For $n=2$:

$$Q^2 = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ -3 & 6 \end{pmatrix} = -3\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$$

$$= -3\,Q^1$$

Antar at gjelder for $n-1$:

$$Q^{n-1} = (-3)^{n-2} Q$$

$$Q^n = Q^{n-1} \cdot Q = (-3)^{n-2} Q^2 = (-3)^{n-2} \cdot (-3) \cdot Q$$

$$= (-3)^{n-1} Q$$

Q.E.D

b) $P(t) = e^{Qt} = \sum_{n=0}^{\infty} \frac{(Qt)^n}{n!} = I + \sum_{n=1}^{\infty} \frac{(-3)^{n-1} Q\, t^n}{n!}$

$= I - \frac{1}{3} Q \sum_{n=1}^{\infty} \frac{(-3t)^n}{n!} = \underline{\underline{I - \frac{1}{3} Q \left(e^{-3t} - 1\right)}}$

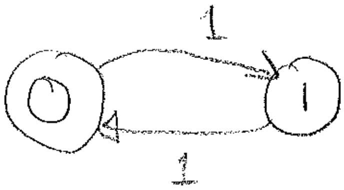
Trekker fra for $n=0$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (s.122 i Hottmann)

c) Q er irredusibel hvis T (overgangsmatrisen for „the embedded Markov chain") er irredusibel (s.183)

$$T = \begin{bmatrix} 0 & -\frac{q_{12}}{q_{22}} \\ -\frac{q_{21}}{q_{11}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$q_{ii} \neq 0$ for $i = 1,2$



T irredusibel $\Rightarrow$ $\underline{\underline{Q \text{ irredusibel}}}$

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \left(I - \frac{1}{3} Q \left(e^{-3t} - 1\right)\right)$$

$$= I + \frac{1}{3} Q = \begin{bmatrix} 1 - \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & 1 - \frac{2}{3} \end{bmatrix} = \underline{\underline{\begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}}}$$

S.16: $\lim_{t \to \infty} P_{ij}(t) = -\frac{1}{q_{ii} \mu_{ii}} \Rightarrow \mu_{ii} = -\frac{1}{q_{ii} \lim_{t \to \infty} P_{ij}(t)}$

$\Rightarrow \mu_{11} = -\frac{1}{(-1) \cdot \frac{2}{3}} = \underline{\underline{\frac{3}{2}}}$

$$\mu_{22} = -\frac{1}{(-2)\cdot\frac{1}{3}} = \underline{\underline{\frac{3}{2}}}$$

d.)

$$Q\pi = \pi$$

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\pi_1 + 2\pi_2 = 0$$

$$\pi_2 = \frac{\pi_1}{2}$$

$$\pi_1 + \pi_2 = \left(1 + \frac{1}{2}\right)\pi_1 = 1$$

$$\Rightarrow \underline{\underline{\pi_1 = \frac{2}{3}}}$$

$$\pi_2 = \frac{1}{2}\cdot\frac{2}{3} = \underline{\underline{\frac{1}{3}}}$$

e.)

$$\pi = \lim_{t\to\infty} P(t)\cdot p(0) = \lim_{t\to\infty} P(t)\cdot\begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix} = \begin{bmatrix} \frac{2}{3}(\underbrace{p_1(0) + p_2(0)}_{=1}) \\ \frac{1}{3}(\underbrace{p_1(0) + p_2(0)}_{=1}) \end{bmatrix} = \underline{\underline{\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}}}$$

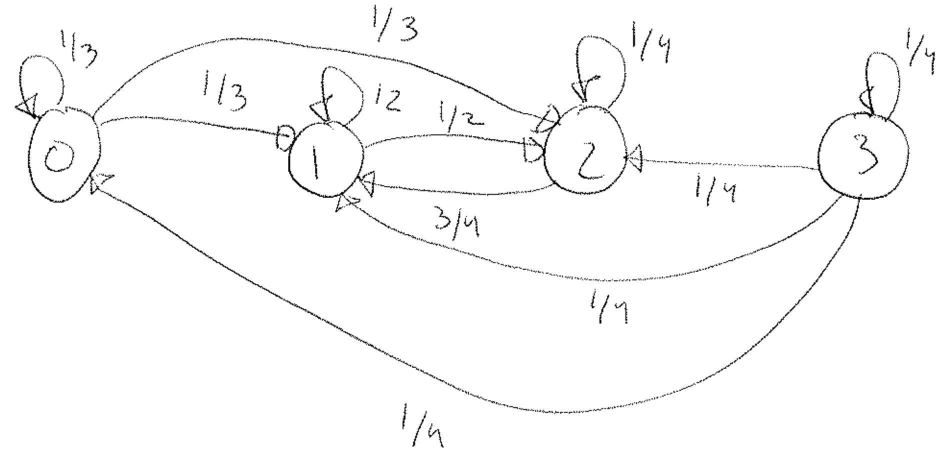
Eksamen i ST114, juni 2000, Oppgave 1 a)-c)

Overgangsmatrise (transponert):

pga "omvendt" måte å definere overgangsmatrisen på i boken som brukes nå.

$$P = \begin{bmatrix} 1/3 & 0 & 0 & \frac{1}{4} \\ 1/3 & 1/2 & 3/4 & 1/4 \\ 1/3 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

a.)

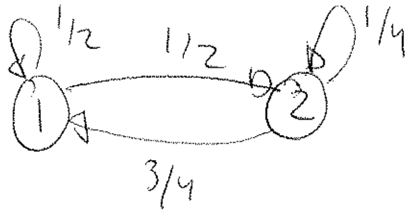


3 klasser: $\{0\}$, $\{1,2\}$, $\{3\}$

$\{0\}$: Transient, $\{1,2\}$: Rekurrent, $\{3\}$: Transient

b.) $\{0\}$ og $\{3\}$ er transiente $\Rightarrow \lim_{n\to\infty} p_{0i}^{(n)} = p_{3i}^{(n)} = 0$

Delkjede:



Endelig, irredusibel, aperiodisk Markovkjede

med overg. matrise:

$$P' = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix} \end{array} \quad \text{og} \quad \lim_{n\to\infty} P'^{(n)} = \begin{bmatrix} \pi_1 & \pi_1 \\ \pi_2 & \pi_2 \end{bmatrix}$$

Finner π_1, π_2:

$$P' \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ 1/2 & -\frac{3}{4} \end{bmatrix} = 0$$

$$\frac{1}{2}\pi_1 = \frac{3}{4}\pi_2$$

$$\pi_1 = \frac{3}{2}\pi_2$$

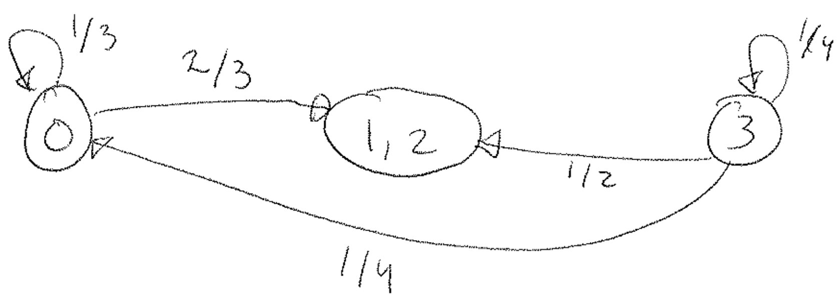
$$\sum_{i=0}^{4} \pi_i = 1 \Rightarrow \frac{5}{2}\pi_2 = 1$$

$$\pi_2 = \frac{2}{5}$$

$$\pi_1 = \frac{3}{5}$$

$$\Rightarrow \lim_{n\to\infty} P^{(n)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3/5 & 3/5 & 3/5 & 3/5 \\ 2/5 & 2/5 & 2/5 & 2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c.) Ser på $\{1,2\}$ som én tilstand, som er rekurrent og absorberende. Har da kjeden



T er tiden til Markovkjeden går til $\{1,2\}$

$v_i = E[T \mid X_0 = i]$ for $i = 0, \{1,2\}, 3$,

Skal finne v_0 og v_3. Har $v_{\{1,2\}} = 0$

$$v_0 = \frac{1}{3}(1 + v_0) + \frac{2}{3} \cdot (1 + \overset{=0}{v_{\{1,2\}}}) = 1 + \frac{1}{3} v_0$$

$$\Rightarrow \underline{\underline{v_0 = \frac{2}{3}}}$$

$$v_3 = \frac{1}{4} \cdot (1 + v_3) + \frac{1}{2} \cdot (1 + \overset{=0}{v_{\{1,2\}}}) + \frac{1}{4} \cdot (1 + v_0)$$

$$= 1 + \frac{1}{4} v_3 + \frac{1}{4} v_0$$

$$\Rightarrow \underline{\underline{v_3 = \frac{11}{6}}}$$