# Ekstraoppgave 9

$X_0 = 1 \quad , \; p_0 = 0 , \quad p_i = \frac{1}{2^i} \quad , i = 1, 2, \ldots$

$|t| < 1$:

$$f(t) = E[t^X] = \sum_{i=0}^{\infty} p_i t^i = \sum_{i=1}^{\infty} \frac{1}{2^i} t^i$$

$$= \sum_{i=1}^{\infty} \left(\frac{t}{2}\right)^i = \sum_{i=0}^{\infty} \left(\frac{t}{2}\right)^i - 1$$

$$= \frac{1}{1 - \frac{t}{2}} - 1 = \underline{\underline{\frac{t}{2-t}}}$$

b.) $h_2(t) = h_1(h_1(t)) = f(f(t)) = \frac{f(t)}{2 - f(t)}$

$$= \frac{\frac{t}{2-t}}{2 - \frac{t}{2-t}} = \frac{t}{2(2-t) - t} = \frac{t}{4-3t} = \underline{\underline{\frac{t}{2^2 - (2^2-1)t}}}$$

$$h_3(t) = h_2(h_1(t)) = \frac{f(t)}{4 - 3 \cdot f(t)} = \frac{\frac{t}{2-t}}{4 - 3 \cdot \frac{t}{2-t}}$$

$$= \frac{t}{4(2-t) - 3t} = \frac{t}{8-7t} = \underline{\underline{\frac{t}{2^3 - (2^3-1)t}}}$$

$$h_4(t) = h_3(h_1(t)) = \frac{f(t)}{8 - 7f(t)} = \frac{\frac{t}{2-t}}{8 - 7 \cdot \frac{t}{2-t}} = \frac{t}{16 - 15t}$$

$$= \frac{t}{2^4-(2^4-1)t}$$

Vet at $h_n(t) = \frac{t}{2^n-(2^n-1)t}$ for $n=1,2,3,4$

Antar $h_{n-1}(t) = \frac{t}{2^{n-1}-(2^{n-1}-1)t}$

$$h_n(t) = h_n(h_1(t)) = \frac{f(t)}{2^{n-1}-(2^{n-1}-1)f(t)}$$

$$= \frac{\frac{t}{2-t}}{2^{n-1}-(2^{n-1}-1)\frac{t}{2-t}} = \frac{t}{2^n-2^{n-1}t-2^{n-1}t+t}$$

$$= \frac{t}{2^n-(2^n-1)t}$$ Q.E.D.

c.) $h_n(t) = \sum_{j=0}^{\infty} P(X_n = j)\, t^j$

Dvs $P(X_n = j)$ er koeffisienten foran $t^j$ i $h_n(t)$

$$h_n(t) = \frac{t}{2^n - (2^n - 1)t} = \frac{t}{2^n} \cdot \frac{1}{1 - (1 - \frac{1}{2^n})t}$$

$$= \frac{t}{2^n} \sum_{j=0}^{\infty} \left(1 - \frac{1}{2^n}\right)^j t^j = \frac{1}{2^n} \sum_{j=0}^{\infty} \left(1 - \frac{1}{2^n}\right)^j t^{j+1}$$

dvs $P(X_n = 0) = 0$

$$P(X_n = j) = \frac{\left(1 - \frac{1}{2^n}\right)^{j-1}}{2^n} \quad \text{for } j > 0$$

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$$E[X_n] = \sum_{j=1}^{\infty} j\, P(X_n = j) = \sum_{j=1}^{\infty} j \frac{\left(1 - \frac{1}{2^n}\right)^{j-1}}{2^n}$$

$$= \frac{1}{2^n\left(1 - \frac{1}{2^n}\right)} \sum_{j=1}^{\infty} j \left(1 - \frac{1}{2^n}\right)^j$$

$$= \frac{1}{2^n\left(1 - \frac{1}{2^n}\right)} \cdot \frac{\left(1 - \frac{1}{2^n}\right)}{\left(1 - \left(1 - \frac{1}{2^n}\right)\right)^2} = \frac{1}{2^n \cdot \left(\frac{1}{2^n}\right)^2}$$

$$= \underline{\underline{2^n}}$$