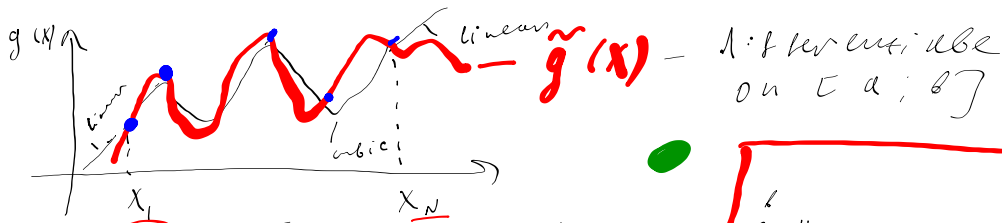


ex 5.7 from the text book 11.03.2022

$\{x_1, z_1\}, \dots, \{x_N, z_N\}, N \geq 2$

$$a < x_1 < \dots < x_N < b$$



a)  $h(x) = \tilde{g}(x) - g(x)$ , show  $\int_a^b g''(x) h''(x) dx = 0$

$\int u dv = uv - \int v du$   
integration by parts

$$\int_a^b g''(x) h''(x) dx = \int_a^b g''(x) d h'(x) =$$

$$= g''(x) h'(x) \Big|_a^b - \int_a^b h'(x) d g''(x) = (*)$$

$(ax + b)'' = (a)' = 0$

$$(*) = - \int_a^b h'(x) g'''(x) dx =$$

$$= - \int_a^{x_N} h'(x) g'''(x) dx =$$

$$= - \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} h'(x) g'''(x) dx =$$

$$= - \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} \underbrace{g'''(x)}_u \underbrace{d h(x)}_{dv} =$$

$$= - \sum_{i=1}^{N-1} g'''(x) h(x) \Big|_{x_i}^{x_{i+1}} + \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} h(x) d g'''(x)$$

$$= \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} h(x) g^{(iv)}(x) dx$$

but  $(ax^3 + bx^2 + cx + d)^{(iv)} = 0$  since

$$= (3ax^2 + 2bx + c)''' = (6ax + 2b)'' =$$

$$= (6a)' = 0$$

$$b) \int_a^b \tilde{g}''(x)^2 dx \geq \int_a^b g''(x)^2 dx$$

$h(x) = \tilde{g}'(x) - g'(x)$   
 $\tilde{g}(x) = h(x) + g(x)$

$$\begin{aligned} \int_a^b \tilde{g}''(x)^2 dx &= \int_a^b [(h'(x) + g'(x))']^2 dx = \\ &= \int_a^b (h''(x)^2 + 2h''(x)g'(x) + (g''(x))^2) dx = \\ &= \int_a^b h''(x)^2 dx + 2 \int_a^b h''(x)g'(x) dx + \int_a^b g''(x)^2 dx \end{aligned}$$

0 from a)

$$= \int_a^b h''(x)^2 dx + \int_a^b g''(x)^2 dx \quad (\Rightarrow)$$

$$\int_a^b \tilde{g}''(x)^2 dx \geq \int_a^b g''(x)^2 dx$$

equality i.i.f. (it and only it)

$$\int_a^b h''(x)^2 dx = 0 \Rightarrow$$

$$\Rightarrow h''(x) = 0 \Rightarrow h'(x) = a \Rightarrow$$

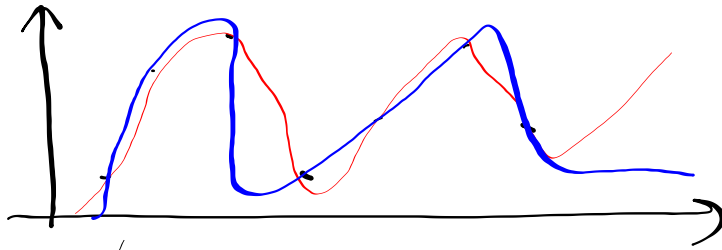
$$\Rightarrow h(x) = ax + b = 0$$

$a$  and  $b$  are  $0 \neq (x)$

$$D.C) \textcircled{(1)} \min_f \left( \sum_{i=1}^N (y_i - f(x_i))^2 + \lambda \int_a^b f''(t)^2 dt \right)$$

$$f_1(x) = \arg \min_{f \in F} \left( \sum_{i=1}^N (y_i - f(x_i))^2 + \lambda \int_a^b f''(t)^2 dt \right)$$

$f_2(x)$  - a natural cubic spline



$$\sum_{i=1}^N (y_i - f_1(x_i))^2 = \sum_{i=1}^N (y_i - f_2(x_i))^2$$

$f_1(x)$  - some function

$f_2(x)$  - natural cubic spline  $\Rightarrow$

$$\Rightarrow \text{from b)} \int_a^b \tilde{y}''(x)^2 dx \geq \int_a^b y''(x)^2 dx$$

in our case this corresponds to

$$\tilde{y}(x) = f_1(x) \quad \text{and} \quad y(x) = f_2(x) \quad \text{hence}$$

$$\int_a^b f_1''(x)^2 dx \geq \int_a^b f_2''(x)^2 dx \Leftrightarrow$$

given that  $f$  must minimize  $\textcircled{(1)}$

$$\int_a^b f_1''(t)^2 dt = \int_a^b f_2''(t)^2 dt \quad \text{and hence}$$

$$f_1(x) = f_2(x) \quad \forall x$$

