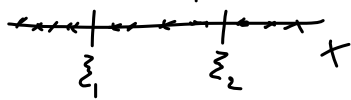


Splines

- global solutions for regression, e.g. linear regression
- local solutions for regression, ^{or} local regression
- piecewise solution:
 - split the support of X into K non-overlapping pieces
 - fit a polynomial on each piece



$K=2$ fix K points $\xi_1, \dots, \xi_K$
with $\xi_1 < \xi_2 < \dots < \xi_K$

Simplest version:

→ fit a constant model on each piece

$$\hat{\beta}_1 \text{ when } x < \xi_1$$

$$\hat{\beta}_2 \text{ when } \xi_1 \leq x < \xi_2$$

$$\hat{\beta}_3 \text{ when } x \geq \xi_2$$

$$\begin{aligned} \hat{f}(x) &= \sum_{m=1}^3 \hat{\beta}_m h_m(x) \\ &= \hat{\beta}_1 \mathbb{1}(x < \xi_1) + \hat{\beta}_2 \mathbb{1}(\xi_1 \leq x < \xi_2) \\ &\quad + \hat{\beta}_3 \mathbb{1}(x \geq \xi_2) \end{aligned}$$

for each piece, we can fit a line

$$\hat{\beta}_1 + \hat{\beta}_4 x \text{ when } x < \xi_1$$

$$\hat{\beta}_2 + \hat{\beta}_5 x \text{ when } \xi_1 \leq x < \xi_2$$

$$\hat{\beta}_3 + \hat{\beta}_6 x \text{ when } x \geq \xi_2$$

} 6 parameters

when we force continuity

$\hat{\beta}_1$ first intercept

$\hat{\beta}_2$ related to the slope

$\hat{\beta}_3$ related to the change in slope after ξ_1

$\hat{\beta}_4$ related to the change in slope after ξ_2

$$h_1(x) = 1$$

$$h_2(x) = x$$

$$h_3(x) = (x - \xi_1)_+$$

$$h_4(x) = (x - \xi_2)_+$$

$$\hat{f}(x) = \hat{\beta}_1 + \hat{\beta}_2 x + \hat{\beta}_3 (x - \xi_1)_+ + \hat{\beta}_4 (x - \xi_2)_+$$

linear regression $X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$

$$H = \begin{pmatrix} h_1(x_1) & h_2(x_1) & \dots & h_n(x_1) \\ h_1(x_2) & h_2(x_2) & \dots & h_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ h_1(x_n) & h_2(x_n) & \dots & h_n(x_n) \end{pmatrix}$$

in this case, with 6 obs

$$H = \begin{pmatrix} 1 & x_1 & x_1 - \xi_1 & x_1 - \xi_2 \\ 1 & x_2 & x_2 - \xi_1 & x_2 - \xi_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_6 & x_6 - \xi_1 & x_6 - \xi_2 \end{pmatrix}$$

$$\hat{f}(x) = \hat{\beta}_1 + \hat{\beta}_2 x + \hat{\beta}_3 x^2 + \hat{\beta}_4 x^3 + \hat{\beta}_5 (x - \xi_1)_+^3 + \hat{\beta}_6 (x - \xi_2)_+^3$$

This is a cubic spline with $k=2$ knots
 of order $M=4$

General formula for splines

$$\hat{f}(x) = \sum_{m=1}^{k+n} \hat{\beta}_m h_m(x)$$

where $h_m(x) = x^{m-1}$ for $m = 1, \dots, M$

$h_{m+K}(x) = (x - \xi_k)^{M-1}$ for $k = 1, \dots, K$