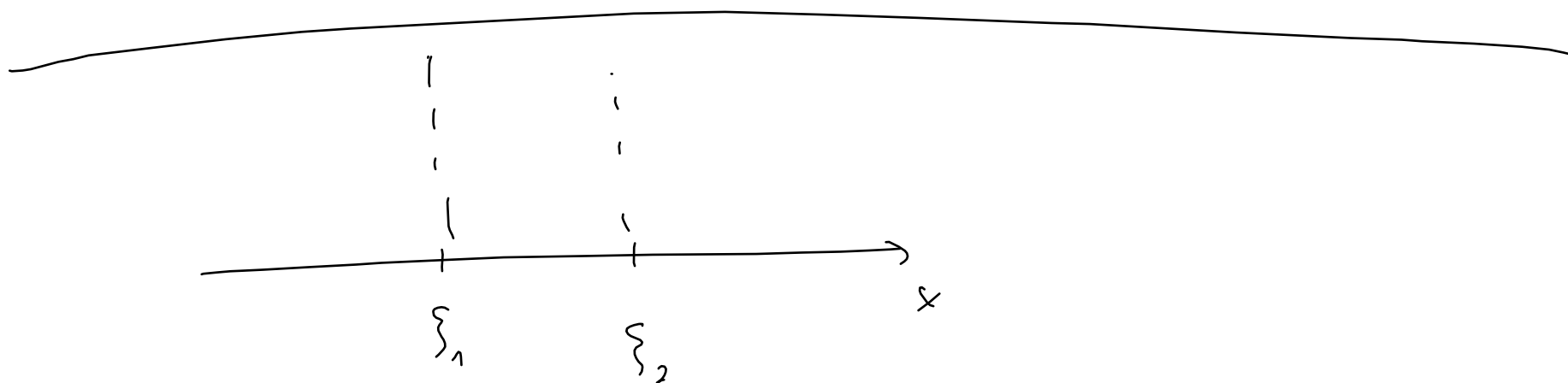


$$Y \in \{0, 1\} \quad , \quad P(Y=1) = p \quad , \quad P(Y=0) = 1-p$$

$$\begin{aligned} E(Y) &= \sum_y y \cdot P(Y=y) = 0 \cdot P(Y=0) + 1 \cdot P(Y=1) \\ &= P(Y=1) = p \end{aligned}$$



$$f(x) = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3 + \sum_{l=1}^K \beta_{4+l} (x - \xi_l)_+^3$$

For  $x < \xi_1$  :

$$f(x) = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3$$

$$\rightarrow f'(x) = \beta_2 + 2\beta_3 x + 3\beta_4 x^2$$

$$\rightarrow f''(x) = 2\beta_3 + 6\beta_4 x$$

$$\rightarrow f^{(3)}(x) = 6\beta_4$$

We get :

$$f''(\xi_1) = 2\beta_3 + 6\beta_4 \xi_1 = 0$$

$$f^{(3)}(\xi_1) = 6\beta_4 = 0 \rightarrow \beta_4 = 0$$

$$\rightarrow f''(\xi_1) = 2\beta_3 = 0 \rightarrow \beta_3 = 0$$

For  $x \geq \xi_K$ :

$$f(x) = \beta_1 + \beta_2 x + \sum_{l=1}^K \beta_{4+l} (x - \xi_l)^3$$

$$\rightarrow f'(x) = \beta_2 + \sum_{l=1}^K 3 \beta_{4+l} (x - \xi_l)^2$$

$$\rightarrow f''(x) = \sum_{l=1}^K 6 \beta_{4+l} (x - \xi_l)$$

$$\rightarrow f^{(3)}(x) = \sum_{l=1}^K 6 \beta_{4+l}$$

We get:

$$\begin{aligned} f''(\xi_K) &= \sum_{l=1}^K 6 \beta_{4+l} (\xi_K - \xi_l) = 6 \sum_{l=1}^{K-1} \beta_{4+l} (\xi_K - \xi_l) + 6 \beta_{4+K} \underbrace{(\xi_K - \xi_K)}_{=0} \\ &= 6 \sum_{l=1}^{K-1} \beta_{4+l} (\xi_K - \xi_l) \end{aligned}$$

$$= 6 \sum_{l=1}^{K-2} \beta_{4+l} (\xi_K - \xi_l) + 6 \beta_{3+K} (\xi_K - \xi_{K-1}) = 0$$

$$\rightarrow \beta_{3+K} = - \sum_{l=1}^{K-2} \beta_{4+l} \frac{\xi_K - \xi_l}{\xi_K - \xi_{K-1}}$$

$$f^{(3)}(\xi_K) = \sum_{l=1}^K 6 \beta_{4+l} = 6 \sum_{l=1}^{K-2} \beta_{4+l} + 6 \beta_{3+K} + 6 \beta_{4+K} = 0$$

$$\begin{aligned} \rightarrow \beta_{4+K} &= -\beta_{3+K} - \sum_{l=1}^{K-2} \beta_{4+l} \\ &= \sum_{l=1}^{K-2} \beta_{4+l} \frac{\xi_K - \xi_l}{\xi_K - \xi_{K-1}} - \sum_{l=1}^{K-2} \beta_{4+l} \\ &= \sum_{l=1}^{K-2} \beta_{4+l} \left( \frac{\xi_K - \xi_l}{\xi_K - \xi_{K-1}} - 1 \right) \\ &= \sum_{l=1}^{K-2} \beta_{4+l} \frac{\xi_K - \xi_l - (\xi_K - \xi_{K-1})}{\xi_K - \xi_{K-1}} \\ &= \sum_{l=1}^{K-2} \beta_{4+l} \frac{\xi_{K-1} - \xi_l}{\xi_K - \xi_{K-1}} \end{aligned}$$

We obtain :

$$\begin{aligned}
 f(x) &= \beta_1 + \beta_2 x + \sum_{l=1}^{K-2} \beta_{4+l} (x - \xi_l)_+^3 + \beta_{3+K} (x - \xi_{K-1})_+^3 + \beta_{4+K} (x - \xi_K)_+^3 \\
 &= \beta_1 + \beta_2 x + \sum_{l=1}^{K-2} \beta_{4+l} (x - \xi_l)_+^3 - \sum_{l=1}^{K-2} \beta_{4+l} \frac{\xi_K - \xi_l}{\xi_K - \xi_{K-1}} (x - \xi_{K-1})_+^3 + \sum_{l=1}^{K-2} \beta_{4+l} \frac{\xi_{K-1} - \xi_l}{\xi_K - \xi_{K-1}} (x - \xi_K)_+^3 \\
 &= \beta_1 + \beta_2 x + \sum_{l=1}^{K-2} \beta_{4+l} \left( (x - \xi_l)_+^3 - (\xi_K - \xi_l) \frac{(x - \xi_{K-1})_+^3}{\xi_K - \xi_{K-1}} + (\xi_{K-1} - \xi_l) \frac{(x - \xi_K)_+^3}{\xi_K - \xi_{K-1}} \right) \\
 &\quad \begin{aligned} &\xi_{K-1} - \xi_l + \xi_K - \xi_K \\ &= (\xi_K - \xi_l) - (\xi_K - \xi_{K-1}) \end{aligned} \\
 &= \beta_1 + \beta_2 x + \sum_{l=1}^{K-2} \beta_{4+l} \left( (x - \xi_l)_+^3 - (\xi_K - \xi_l) \frac{(x - \xi_{K-1})_+^3}{\xi_K - \xi_{K-1}} + (\xi_K - \xi_l) \frac{(x - \xi_K)_+^3}{\xi_K - \xi_{K-1}} - \cancel{(\xi_K - \xi_{K-1}) \frac{(x - \xi_l)_+^3}{\xi_K - \xi_{K-1}}} \right) \\
 &= \beta_1 + \beta_2 x + \sum_{l=1}^{K-2} \underbrace{\beta_{4+l} (\xi_K - \xi_l)}_{\tilde{\beta}_{2+l}} \left( \underbrace{\frac{(x - \xi_l)_+^3 - (x - \xi_K)_+^3}{\xi_K - \xi_l}}_{d_l(x)} - \underbrace{\frac{(x - \xi_{K-1})_+^3 - (x - \xi_K)_+^3}{\xi_K - \xi_{K-1}}}_{d_{K-1}(x)} \right) \\
 &= \beta_1 + \beta_2 x + \sum_{l=1}^{K-2} \tilde{\beta}_{2+l} \underbrace{(d_l(x) - d_{K-1}(x))}_{N_{2+l}(x)} = \beta_1 + \beta_2 x + \sum_{l=1}^{K-2} \tilde{\beta}_{2+l} N_{2+l}(x)
 \end{aligned}$$