

We have:

$$\hat{y} = \operatorname{argmax}_k P(Y=k | X=x)$$

$$= \operatorname{argmax}_k \frac{\pi_k p_k(x)}{\sum_{i=1}^K \pi_i p_i(x)}$$

$$= \operatorname{argmax}_k \pi_k p_k(x)$$

$$= \operatorname{argmax}_k \left(\pi_k \cdot \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu_k)^T \Sigma^{-1}(x-\mu_k)} \right)$$

$$= \operatorname{argmax}_k \log \left(\pi_k \cdot \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu_k)^T \Sigma^{-1}(x-\mu_k)} \right)$$

$$= \operatorname{argmax}_k \left(\log(\pi_k) - \frac{p}{2} \log(2\pi) - \frac{1}{2} \log(|\Sigma|) - \frac{1}{2}(x-\mu_k)^T \Sigma^{-1}(x-\mu_k) \right)$$

$$= \operatorname{argmax}_k \left(\log(\pi_k) - \frac{1}{2} x^T \Sigma^{-1} x + \underbrace{\frac{1}{2} x^T \Sigma^{-1} \mu_k + \frac{1}{2} \mu_k^T \Sigma^{-1} x}_{= \frac{1}{2} \mu_k^T \Sigma^{-1} x} - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k \right)$$

$$= \operatorname{argmax}_k \left(\underbrace{\mu_k^T \Sigma^{-1} x + \log(\pi_k) - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k}_{\delta_k(x)} \right)$$

$$= \operatorname{argmax}_k \delta_k(x)$$

For the LDA, we have:

$$P(Y=1|x) = \frac{\pi_1 p_1(x)}{\pi_1 p_1(x) + (1-\pi_1) p_0(x)}$$

$$P(Y=0|x) = \frac{(1-\pi_1) p_0(x)}{\pi_1 p_1(x) + (1-\pi_1) p_0(x)}$$

We get:

$$\log \frac{P(Y=1|x)}{P(Y=0|x)} = \log \frac{\frac{\pi_1 p_1(x)}{\pi_1 p_1(x) + (1-\pi_1) p_0(x)}}{\frac{(1-\pi_1) p_0(x)}{\pi_1 p_1(x) + (1-\pi_1) p_0(x)}}$$

$$= \log \frac{\pi_1 p_1(x)}{(1-\pi_1) p_0(x)}$$

$$= \log(\pi_1) - \log(\pi_0) + \log(p_1(x)) - \log(p_0(x))$$

$$= \log(\pi_1) - \log(\pi_0) - \frac{1}{2} \log(2\pi) - \frac{1}{2} \log(|\Sigma|) - \frac{1}{2} (\tilde{x} - \mu_1)^T \Sigma^{-1} (\tilde{x} - \mu_1) - \left(-\frac{1}{2} \log(2\pi) - \frac{1}{2} \log(|\Sigma|) - \frac{1}{2} (\tilde{x} - \mu_0)^T \Sigma^{-1} (\tilde{x} - \mu_0) \right)$$

$$= \log(\pi_1) - \log(\pi_0) - \frac{1}{2} \tilde{x}^T \Sigma^{-1} \tilde{x} + \mu_1^T \Sigma^{-1} \tilde{x} - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2} \tilde{x}^T \Sigma^{-1} \tilde{x} - \mu_0^T \Sigma^{-1} \tilde{x} + \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0$$

$$= \log(\pi_1) - \log(\pi_0) - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0 + \underbrace{(\mu_1^T \Sigma^{-1} - \mu_0^T \Sigma^{-1})}_{\alpha^T} \tilde{x}$$

$= \alpha_0$

$$= \alpha_0 + \alpha^T \tilde{x} = \alpha_0 + \tilde{x}^T \alpha$$