

ESL Ex 4.1

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Show how to solve

$$\begin{aligned} \max_a \quad & \overset{1 \times p}{a^T} \overset{p \times p}{B} \overset{p \times 1}{a} = \\ \text{s.t.} \quad & a^T W a = 1 \end{aligned}$$

by transforming the problem to a standard eigen value problem.

$$\overset{p \times p}{C} \overset{p \times 1}{a} = \lambda \overset{p \times 1}{a}$$

→ a is the eigenvector of C
 λ the corresponding eigenvalue

Note:

$$\begin{aligned} a^T B a &= [a_1 \dots a_p] \begin{bmatrix} B_{11} & \dots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{p1} & \dots & B_{pp} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} \\ &= [a_1 \dots a_p] \begin{bmatrix} \sum_{j=1}^p B_{1j} a_j \\ \vdots \\ \sum_{j=1}^p B_{pj} a_j \end{bmatrix} \\ &= \sum_{i=1}^p \sum_{j=1}^p a_i a_j B_{ij} \end{aligned}$$

Solution (by Weatherwax & Epstein):

Define the Lagrangian:

$$\begin{aligned} \mathcal{L}(a, \lambda) &= a^T B a - \lambda (a^T W a - 1) \\ &= \sum_i \sum_j a_i a_j B_{ij} - \lambda \left[\sum_i \sum_j a_i a_j W_{ij} - 1 \right] \end{aligned}$$

$$\frac{\partial u^T A v}{\partial x} = \frac{\partial u}{\partial x} A v + \frac{\partial v}{\partial x} A^T u$$

$$\frac{\partial x^T A x}{\partial x} = A x + A^T x = (A + A^T)x$$

matrix
derivative
rules
(from ')

First order condition:

$$\frac{\partial \mathcal{L}(a, \lambda)}{\partial a} = \frac{\partial a^T B a}{\partial a} - \lambda \frac{\partial a^T W a}{\partial a}$$

$$= (B + B^T) a - \lambda (W + W^T) a$$

$$= B' a - \lambda W' a$$

where $B' = (B + B^T)$ and $W' = (W + W^T)$

For $\mathcal{L}(a, \lambda) / \partial a = 0$, we get

$$B' a = \lambda W' a$$

Pre-multiply with $(W')^{-1}$ on both sides,
we obtain the standard eigenvalue formulation

$$\underbrace{(W')^{-1} B'}_C a = \lambda a$$

