

The set of basis functions in eq 7.9 are:

$$\begin{aligned}
 h_1(x) &= 1 \\
 h_2(x) &= x \\
 h_3(x) &= x^2 \\
 h_4(x) &= x^3 \\
 h_5(x) &= (x - \xi_1)_+^3 \\
 h_6(x) &= (x - \xi_2)_+^3
 \end{aligned}
 \quad
 \left.
 \begin{aligned}
 h_j(x) &= x^{j-1}, & j=1, 2, \dots, M=4 \\
 h_{n+l}(x) &= (x - \xi_l)_+^{n-1}, & l=1, \dots, K=2
 \end{aligned}
 \right\}
 \text{truncated-power basis functions}$$

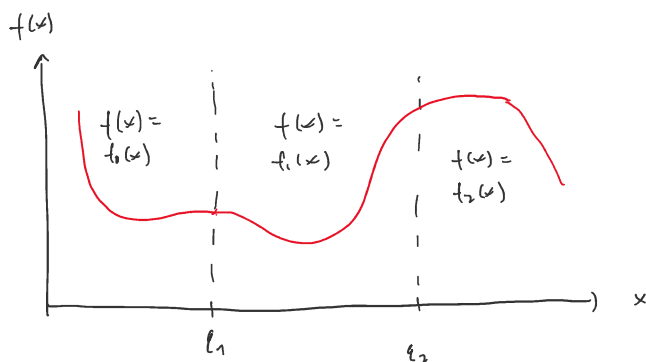
The associated basis-model:

$$\begin{aligned}
 f(x) &= \sum_{j=1}^{M+K} \beta_j h_j(x) \\
 &= \sum_{j=1}^M \beta_j x^{j-1} + \sum_{l=1}^K \beta_{n+l} (x - \xi_l)_+^{n-1}
 \end{aligned}
 \quad (*)$$

with $M=4$ and $K=2$.

We are asked to show that the basis model (*) represents a cubic spline.

A cubic spline is a piecewise third-order (cubic) polynomial with continuous first and second order derivatives.



Thus we need to show that we can rewrite (*) as

$$f(x) = \begin{cases} f_0(x) & \text{if } x < \xi_1 \\ f_1(x) & \text{if } \xi_1 \leq x < \xi_2 \\ f_2(x) & \text{if } \xi_2 \leq x \end{cases} \quad (**)$$

where each $f_i(x)$ is a cubic polynomial:

$$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i,$$

such that the following continuity restrictions are satisfied:

$$f_{i-1}(\xi_i) = f_i(\xi_i)$$

$$f'_{i-1}(\xi_i) = f'_i(\xi_i)$$

$$f''_{i-1}(\xi_i) = f''_i(\xi_i)$$

} these restrict the coefficients a_i, b_i, c_i, d_i

(i) Show that (**) defines a set of cubic polynomials:

For $x < \xi_1$:

$$f(x) = \underbrace{\beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3}_{\text{a cubic polynomial}} + \underbrace{\sum_{k=1}^{\infty} \beta_{4+k} (x - \xi_1)^{k-1}}_{=0}$$

$$\equiv f_0(x)$$

For $\xi_1 \leq x < \xi_2$:

$$f(x) = f_0(x) + \beta_{4+1} (x - \xi_1)^3 + 0$$

$$\begin{aligned} (x - \xi_1)^3 &= x^3 - 3\xi_1 x^2 + 3\xi_1^2 x - \xi_1^3 \\ &= (\beta_1 - \xi_1^3) + (\beta_2 + 3\beta_{4+1}\xi_1^2)x + (\beta_3 - 3\beta_{4+1}\xi_1)x^2 + (\beta_4 + \beta_{4+1})x^3 \end{aligned}$$

$$= d_1 + c_1 x + b_1 x^2 + a_1 x^3$$

$$= f_1(x)$$

in
a cubic polynomial

For $x \geq \xi_2$:

$$f(x) = f_1(x) + \underbrace{\beta_{4+2} (x - \xi_2)^3}_{\text{also a cubic polynomial}} = f_2(x)$$

Hence, we can indeed rewrite $f(x)$ as a piecewise polynomial:

$$f(x) = \sum_{j=1}^{4+n} \beta_j h_j(x) = \begin{cases} f_0(x) & x < \xi_1 \\ f_1(x) & \xi_1 \leq x < \xi_2 \\ f_2(x) & x \geq \xi_2 \end{cases}$$

Where

$$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i = f_{i-1}(x) + \beta_{4+i} (x - \xi_i)^3$$

which implies

$$d_i = d_{i-1} - \beta_{4+i} \xi_i^3$$

$$c_i = c_{i-1} + 3 \beta_{4+i} \xi_i^2$$

$$b_i = b_{i-1} - 3 \beta_{4+i} \xi_i$$

$$a_i = a_{i-1} - \beta_{4+i}$$

$$\Leftrightarrow \beta_{4+i} = -(a_i - a_{i-1})$$

(ii) Show that $f(x)$, $f'(x)$ and $f''(x)$ are continuous in each knot

This follows from

$$h_{m,i}(\xi_i) = h'_{m,i}(\xi_i) = h''_{m,i}(\xi_i) = 0$$

which implies that for all knots ξ_i :

$$f_i(\xi_i) = f_{i-1}(\xi_i) + \beta_{m,i}(\xi_i - \xi_i)^3 = f_{i-1}(\xi_i)$$

$$f'_i(\xi_i) = f'_{i-1}(\xi_i) + 3\beta_{m,i}(\xi_i - \xi_i)^2 = f'_{i-1}(\xi_i)$$

$$f''_i(\xi_i) = f''_{i-1}(\xi_i) + 6\beta_{m,i}(\xi_i - \xi_i) = f''_{i-1}(\xi_i)$$