

We have :

$$\begin{aligned}
 T &= \frac{1}{2} \sum_{i=1}^n \sum_{i'=1}^n d_{ii'} \quad , \quad d_{ii'} = d(\underline{x}_i, \underline{x}_{i'}) \\
 &= \frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \left(\sum_{\{i': C(i')=k\}} d_{ii'} + \sum_{\{i': C(i') \neq k\}} d_{ii'} \right) \\
 &= \underbrace{\frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} d_{ii'}}_{W(C)} + \underbrace{\frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i') \neq k\}} d_{ii'}}_{B(C)}
 \end{aligned}$$

We have:

$$\begin{aligned}
 W(C) &= \frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} d_{ii'} \\
 &= \frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} \|\underline{x}_i - \underline{x}_{i'}\|^2 \\
 &= \frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} (\underline{x}_i - \underline{x}_{i'})^T (\underline{x}_i - \underline{x}_{i'}) \\
 &= \frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} (\underline{x}_i - \bar{\underline{x}}_k + \bar{\underline{x}}_k - \underline{x}_{i'})^T (\underline{x}_i - \bar{\underline{x}}_k + \bar{\underline{x}}_k - \underline{x}_{i'}) \\
 &= \frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} \left((\underline{x}_i - \bar{\underline{x}}_k)^T (\underline{x}_i - \bar{\underline{x}}_k) + \underbrace{(\underline{x}_{i'} - \bar{\underline{x}}_k)^T (\underline{x}_i - \bar{\underline{x}}_k)}_{(\bar{\underline{x}}_k - \underline{x}_{i'})^T (\bar{\underline{x}}_k - \underline{x}_{i'})} \right. \\
 &\quad \left. + 2(\underline{x}_i - \bar{\underline{x}}_k)^T (\bar{\underline{x}}_k - \underline{x}_{i'}) \right) \\
 &= \frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} (\underline{x}_i - \bar{\underline{x}}_k)^T (\underline{x}_i - \bar{\underline{x}}_k) + \frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} (\underline{x}_{i'} - \bar{\underline{x}}_k)^T (\underline{x}_{i'} - \bar{\underline{x}}_k) \\
 &\quad + \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} (\underline{x}_i - \bar{\underline{x}}_k)^T (\bar{\underline{x}}_k - \underline{x}_{i'})
 \end{aligned}$$

We have

$$\sum_{k=1}^K \sum_{\{i: C(i)=k\}} (\underline{x}_i - \bar{\underline{x}}_k)^T \sum_{\{i': C(i')=k\}} (\bar{\underline{x}}_k - \underline{x}_{i'})$$

where

$$\sum_{\{i: C(i)=k\}} (\bar{\underline{x}}_k - \underline{x}_{i'}) = \sum_{\{i: C(i)=k\}} \bar{\underline{x}}_k - \sum_{\{i: C(i)=k\}} \underline{x}_{i'} = 0$$

$N_k \cdot \bar{\underline{x}}_k = N_k \cdot \frac{1}{N_k} \sum_{\{i: C(i)=k\}} \underline{x}_{i'}$

with $N_k = \sum_{i=1}^N \mathbb{I}(C(i)=k)$, i.e. the number of points in cluster k

We get:

$$\begin{aligned}
 W(C) &= \frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} (\underline{x}_i - \bar{\underline{x}}_k)^T (\underline{x}_i - \bar{\underline{x}}_k) \\
 &\quad + \frac{1}{2} \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \sum_{\{i': C(i')=k\}} (\underline{x}_{i'} - \bar{\underline{x}}_k)^T (\underline{x}_{i'} - \bar{\underline{x}}_k) \\
 &\quad + \sum_{k=1}^K \sum_{\{i: C(i)=k\}} \underbrace{N_k (\underline{x}_i - \bar{\underline{x}}_k)^T (\underline{x}_i - \bar{\underline{x}}_k)}_{\|\underline{x}_i - \bar{\underline{x}}_k\|^2} \\
 &= \sum_{k=1}^K \sum_{\{i: C(i)=k\}} N_k \underbrace{(\underline{x}_i - \bar{\underline{x}}_k)^T (\underline{x}_i - \bar{\underline{x}}_k)}_{\|\underline{x}_i - \bar{\underline{x}}_k\|^2} \\
 &= \sum_{k=1}^K \sum_{\{i: C(i)=k\}} N_k \|\underline{x}_i - \bar{\underline{x}}_k\|^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } g(\underline{m}) &= \sum_{i \in S} \|\underline{x}_i - \underline{m}\|^2 \\
 &= \sum_{i \in S} (\underline{x}_i - \underline{m})^T (\underline{x}_i - \underline{m}) \\
 &= \sum_{i \in S} \underline{x}_i^T \underline{x}_i - 2 \sum_{i \in S} \underline{x}_i^T \underline{m} + \underbrace{\sum_{i \in S} \underline{m}^T \underline{m}}_{= N_s \underline{m}^T \underline{m}} \\
 &= N_s \underline{m}^T \underline{m}
 \end{aligned}$$

We have:

$$\begin{aligned}
 \frac{\partial g(\underline{m})}{\partial \underline{m}} &= -2 \sum_{i \in S} \underline{x}_i + N_s \underbrace{\left(\underline{\mathbb{I}} + \underline{\mathbb{I}}^T \right)}_{2 \underline{\mathbb{I}}} \underline{m} \\
 &= -2 \sum_{i \in S} \underline{x}_i + 2 N_s \underline{m} = 0 \\
 &\rightarrow N_s \underline{m} = \sum_{i \in S} \underline{x}_i \\
 &\rightarrow \underline{m} = \frac{1}{N_s} \sum_{i \in S} \underline{x}_i = \underline{\bar{x}}_s
 \end{aligned}$$