

We have for the logistic regression model:

$$\log \frac{P(Y=1|x)}{P(Y=0|x)} = \log(e^{\beta_0 + \tilde{x}^T \tilde{\beta}}) = \beta_0 + \tilde{x}^T \tilde{\beta}$$

For the LDA, we have:

$$P(Y=1|x) = \frac{\pi_1 p_1(\tilde{x})}{\pi_1 p_1(\tilde{x}) + \underbrace{\pi_0}_{=1-\pi_1} p_0(\tilde{x})} = \frac{\pi_1 p_1(\tilde{x})}{\pi_1 p_1(\tilde{x}) + (1-\pi_1) p_0(\tilde{x})}$$

$$P(Y=0|x) = \frac{\pi_0 p_0(\tilde{x})}{\pi_1 p_1(\tilde{x}) + \pi_0 p_0(\tilde{x})} = \frac{(1-\pi_1) p_0(\tilde{x})}{\pi_1 p_1(\tilde{x}) + (1-\pi_1) p_0(\tilde{x})}$$

We get:

$$\log \frac{P(Y=1|x)}{P(Y=0|x)} = \log \frac{\frac{\pi_1 p_1(\tilde{x})}{\pi_1 p_1(\tilde{x}) + (1-\pi_1) p_0(\tilde{x})}}{\frac{(1-\pi_1) p_0(\tilde{x})}{\pi_1 p_1(\tilde{x}) + (1-\pi_1) p_0(\tilde{x})}}$$

$$= \log \frac{\pi_1 p_1(\tilde{x})}{(1-\pi_1) p_0(\tilde{x})}$$

$$= \log(\pi_1) - \log(1-\pi_1) + \log p_1(\tilde{x}) - \log p_0(\tilde{x})$$

$$= \log(\pi_1) - \log(1-\pi_1) + \left(-\frac{1}{2} \log(|\Sigma|) - \frac{1}{2} (\tilde{x} - \mu_1)^T \Sigma^{-1} (\tilde{x} - \mu_1) \right) - \left(-\frac{1}{2} \log(|\Sigma|) - \frac{1}{2} (\tilde{x} - \mu_0)^T \Sigma^{-1} (\tilde{x} - \mu_0) \right)$$

$$= \log(\pi_1) - \log(1-\pi_1) - \frac{1}{2} \left(\tilde{x}^T \Sigma^{-1} \tilde{x} - 2 \tilde{x}^T \Sigma^{-1} \mu_1 + \mu_1^T \Sigma^{-1} \mu_1 \right) + \frac{1}{2} \left(\tilde{x}^T \Sigma^{-1} \tilde{x} - 2 \tilde{x}^T \Sigma^{-1} \mu_0 + \mu_0^T \Sigma^{-1} \mu_0 \right)$$

$$= \underbrace{\tilde{x}^T \left(\Sigma^{-1} \mu_1 - \Sigma^{-1} \mu_0 \right)}_{= \underline{\alpha}} + \underbrace{\log(\pi_1) - \log(1-\pi_1) - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2} \mu_0^T \Sigma^{-1} \mu_0}_{= \alpha_0}$$

$$= \alpha_0 + \tilde{x}^T \underline{\alpha}$$