STK2130: Solution to Exam Spring 2007

Problem 2

Facebook exercise and keeping notations:

(a)

$$q_{i,i+1} = \lim_{\Delta t \to 0} \frac{p_{i,i+1}(\Delta t) - p_{i,i+1}(0)}{\Delta t} = \lambda_i,$$

$$q_{i,i-1} = \lim_{\Delta t \to 0} \frac{p_{i,i-1}(\Delta t) - p_{i,i-1}(0)}{\Delta t} = \mu_i,$$

$$q_{i,i} = \lim_{\Delta t \to 0} \frac{p_{i,i}(\Delta t) - p_{i,i}(0)}{\Delta t} = -(\lambda_i + \mu_i).$$

The infinitesimal generator matrix is $Q = (q_{ij})_{ij \in S}$ defined as $Q := P'(0)$ which generates $P(t)$ in the following way

$$P(t) = e^{Qt}.$$

The limiting probabilities $\pi = (\pi_0, \pi_1, \dots)$ are found as the solution to

$$\pi Q = 0$$

and such that

$$\sum_{i=0}^{\infty} \pi_i = 1.$$

Now

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \dots & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \dots & 0 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots \end{pmatrix}$$

Then $\pi Q = 0$ gives

$$-\lambda_0 \pi_0 + \mu_1 \pi_1 = 0$$

and for $i \geqslant 1$

$$\lambda_{i-1}\pi_{i-1} - (\lambda_i + \mu_i)\pi_i + \mu_{i+1}\pi_{i+1} = 0. \tag{0.1}$$

We see that $\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$, then using (0.1) we get $\pi_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2}\pi_0$ and so on. Thus

$$\pi_i = \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j}\pi_0$$

provided $\lambda_0 > 0$ and the sum of π_i's is convergent.

(b) Assume that such user joins a new group according to a Poisson process with rate λ and for each group he is a member of decides to leave it with a rate μ (i.e., the chance of leaving it inside $[t, t+\Delta t]$ given that he is a member at time t, is $\mu\Delta t + o(\Delta t)$). Why these assumption imply that the model (2) is in force with

$$\lambda_i = \lambda \text{ and } \mu_i = i\mu, \text{ for } i = 0, 1, 2, \dots$$

Denote by $N(t)$ the number of groups that have been added until time t (without counting the ones left). Then $N(t)$ (see page 312 and 314 for the two equivalent definitions) is a counting process moreover, since "arrivals" or "joins" happen at a Poisson speed, then N is a Poisson distribution (it has stationary independent increments, the book usually assumes this for any markov process p372). We know that since N is Poisson

$$P(N(\Delta t) = 1) = \lambda \Delta t + o(\Delta t)$$

(this is saying that we join a new group in a small time interval with rate λ, in the definition we also assume that two or more arrivals can not happen in a short time interval: see property (iv) page 314))

On the other hand, the probabiity of joining a group is given by

$$p_{i,i+1} = P(X(t+\Delta t) = i+1 | X(t) = i) = \lambda_i \Delta t + o(\Delta t)$$

and by stationary increments we obtain $\lambda_i = \lambda$.

Finally, given we have $X(t) = i$ groups, and each of them are left at a rate of μ and therefore (by independence of groups) $\mu_i = i\mu$ for all $i \geqslant 1$.

(c) From the fact that

$$\pi_i = \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j} \pi_0 = \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \pi_0$$

and

$$\sum_{i=0}^{\infty} \pi_i = 1$$

we see that

$$\pi_0 = \left(\sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \right)^{-1} = e^{-\lambda/\mu}$$

and therefore for each $i \geqslant 0$

$$\pi_i = \frac{e^{-\lambda/\mu}(\lambda/\mu)^i}{i!}.$$

We see clearly that $\pi \sim Poiss(\lambda/\mu)$ and hence the mean is

$$E\pi = \lambda/\mu.$$

(Here we make an abuse of notation and denote by π the asymptotic distribution as we usually do)

(d) $m(t) = EX(t) = E[X(t)|X(0)]$ and recall $X(0) = 0$ which will determine the initial condition of the ODE for $m(t)$.

$E[X(t+\Delta)|X(t) = i] = (X(t)+1)\lambda_i\Delta t+(X(t)-1)\mu_i\Delta t+X(t)\left(1-(\lambda_i+\mu_i)\Delta t+o(\Delta t)\right).$

Now, since $\lambda_i = \lambda$, $\mu_i = i\mu$ and it is given that $i = X(t)$ we have

$E[X(t+\Delta t)|X(t) = i] = (X(t)+1)\lambda\Delta t+(X(t)-1)X(t)\mu\Delta t+X(t)\left(1-(\lambda+X(t)\mu)\Delta t+o(\Delta t)\right).$

We can ignore the terms in $o(\Delta t)$ and then

$$E[X(t+\Delta t)|X(t) = i] = \lambda\Delta t - X(t)\mu\Delta t + X(t).$$

Then

$$m'(t) = \lim_{\Delta t\to 0}\frac{\lambda\Delta t - \mu m(t)\Delta t + m(t) - m(t)}{\Delta t} = \lambda - \mu m(t).$$

The differential equation to solve is

$$m'(t) + \mu m(t) = \lambda$$

$$m(0) = 0$$

How to solve it! (not important) A clear particular solution is $m_p(t) = \lambda/\mu$. The homogeneous solution is $m_h(t) = Ce^{-\mu t}$. So the general solution is

$$m_g(t) = Ce^{-\mu t} + \frac{\lambda}{\mu}.$$

Imposing $m(0) = 0$ we obtain

$$m(t) = \frac{\lambda}{\mu}\left(1 - e^{-\mu t}\right).$$

Another way to justify $m(t) = \frac{\lambda}{\mu}(1-e^{-\mu t})$ as a solution is that linear ODE's with constant coefficients always admit a unique solution, if the one they give us satisfies the equation, which it does indeed, then it IS the solution.

(e) We add a ceiling for the total number of groups we wish to be a member of. The rates turn out to be

$$\lambda_i = (N-i)\lambda \quad \mu_i = i\mu \quad i = 0, \ldots, N.$$

We easily see that

$$\pi_i = \frac{\lambda_0\cdots\lambda_{i-1}}{\mu_1\cdots\mu_i}\pi_0 = \frac{N(N-1)\cdots(N-i-1)}{i!}\left(\frac{\lambda}{\mu}\right)^i\pi_0 = \binom{N}{i}\left(\frac{\lambda}{\mu}\right)^i\pi_0$$

Then we have $\sum_{i=0}^{N}\pi_i = 1$ we deduce

$$\pi_i = \frac{\binom{N}{i}\left(\frac{\lambda}{\mu}\right)^i}{\sum_{j=0}^{N}\binom{N}{j}\left(\frac{\lambda}{\mu}\right)^j}.$$

Since $\sum_{j=0}^{N} \binom{N}{j} \left(\frac{\lambda}{\mu}\right)^j = (1+\frac{\lambda}{\mu})^N$ we can write

$$\pi_i = \binom{N}{i} \left(\frac{\lambda}{\lambda+\mu}\right)^i \left(\frac{\mu}{\lambda+\mu}\right)^{N-i}$$

thus being π binomial distributed with parameters N and $p = \frac{\lambda}{\lambda+\mu}$. See page 394 too.

(f) The previous study of the process $X(t)$ is meant to be for a single user. Now take a population of users where each one has his own join/leave-rate λ/μ. Assume $\lambda/\mu \sim exp(0.1)$. Let X be the number of groups of a randomly selected user of this population where we assume users have reached an equilibirum. Compute the mean and variance of X.

Note that

$$E[X|\lambda/\mu] = \lambda/\mu$$

because given the "join/leave"-rate we know that (in equilibrium) a user is member of λ/μ groups in mean. By the "tower property" we have

$$E[X] = E[\lambda/\mu] = 1/0.1 = 10.$$

Finally by the total variance formula we have

$$\begin{aligned} Var(X) &= E[Var(X|\lambda/\mu)] + Var(E[X|\lambda/\mu]) \\ &= E[\lambda/\mu] + Var(\lambda/\mu) \\ &= 10 + \frac{1}{0.1^2} = 110. \end{aligned}$$

Problem 4

(a) $s \leqslant t$ then

$$E[W(t)W(s)] = E[(W(t) - W(s))W(s) + W(s)^2] = E[W(s)^2] = \sigma^2 s.$$

(b) Define $Z(t) := \frac{W(t)}{\sqrt{t}}$.

$$Var[Z(t)] = E\left[W(t)^2/t\right] - E\left[W(t)/\sqrt{t}\right]^2 = \frac{1}{t}E[W(t)^2] = \frac{1}{t}\sigma^2 t = \sigma^2.$$

Now if $s \leqslant t$

$$Cov[Z(t), Z(s)] = E[Z(t)Z(s)] = \frac{1}{\sqrt{ts}}E[W(t)W(s)] = \frac{\sigma^2 s}{\sqrt{ts}}.$$

The correlation between $Z(t)$ and $Z(s)$ is then

$$\rho_{Z(t),Z(s)} = \frac{Cov[Z(t), Z(s)]}{\sqrt{Var[Z(t)]}\sqrt{Var[Z(s)]}} = \frac{\sigma^2 s/\sqrt{ts}}{\sigma^2} = \frac{s}{\sqrt{ts}}.$$