STK2130: Solution to Exam Spring 2004

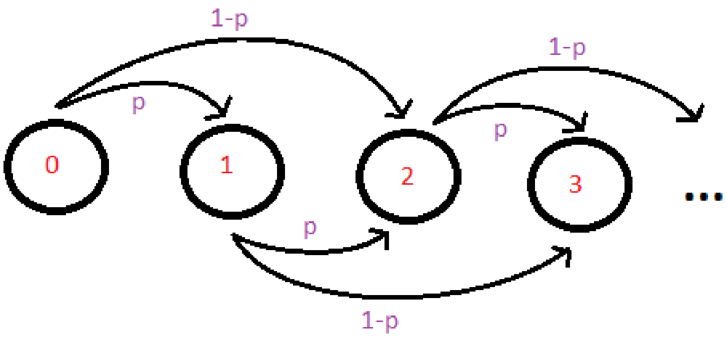
Problem 1

Per's problem.

(a) See book. Take two states that communicate $i \leftrightarrow j$ (i.e. they belong to the same class) and i is recurrent, then we will come back to i at a later time. Since $i \to j$ and i recurrent this implies $j \to i$ and hence $(i \leftrightarrow j)$ j will be recurrent as well.

(b) The states are $S = \{0, 1, \ldots, M\}$. All states are transient (Per goes up but never down) except from M which is recurrent, once he has reached step M he "remains" there. There are $M + 1$ classes since no states communicate. States have no period (not even 1) since once Per is in state i he will (with probability 1) never reenter state i, this only happens in state M where he remains and therefore state M has period 1.

(c) What is the probability that Per goes up the staircase without treading on step no 2?



Here we see that

$$P(X_n \neq 2 | X_0 = 0) = p(1-p)$$

$$P(X_n \neq 2 | X_0 = 1) = 1 - p$$

$$P(X_n \neq 2 | X_0 = 2) = 0$$

$$P(X_n \neq 2 | X_0 \geqslant 3) = 1.$$

We use now the law of total probability and of course, assume that $X_0 = 0$ (Per starts going up from the first floor, i.e. $P(X_0 = 0) = 1$ and $P(X_0 \geqslant 1) = 0$). So

$$\begin{aligned} P(X_n \neq 2) &= P(X_n \neq 2 | X_0 = 0)P(X_0 = 0) + P(X_n \neq 2 | X_0 = 1)P(X_0 = 1) \\ &\quad + P(X_n \neq 2 | X_0 \geqslant 3)P(X_0 \geqslant 3) \\ &= p(1-p). \end{aligned}$$

(d) You shall find the probability that Per goes up the staircase without treading on step no $M - 1$ (because it is squeaking). Introduce the notation u_i for the probability that Per goes up the staircase without treading on step no $M-1$ given he is standing on step no i.

$$u_i = P(X_n \neq M-1, n \geqslant 1 | X_0 = i).$$

It is easy to see the following: (by the law of total probability) given Per is standing on step no i, then he has two possibilities to move along and therefore

$$P(X_n \neq M-1 | X_0 = i) = P(X_n \neq M-1 | X_0 = i+1)p + P(X_n \neq M-1 | X_0 = i+2)(1-p)$$

Now, by the Markov property we conclude

$$u_{i-1} = u_i p + u_{i+1}(1-p) \quad \text{for each} \quad i = 1, \ldots, M-1. \tag{0.1}$$

In particular, it is also easy to compute u_i for i close to $M-1$. Namely,

$$u_{M-2} = 1-p$$

$$u_{M-3} = p(1-p)$$

$$u_{M-4} = p^2(1-p) + (1-p)^2$$

$$u_{M-5} = p^3(1-p) + p(1-p)^2 + p(1-p)^2 = p^3(1-p) + 2p(1-p)^2$$

and one sees that they satisfy (0.1)

(e) The difference equation given in (0.1) can be solved and for $i = 0$ (Per does not tread on step $M-1$) is

$$u_0 = \frac{q + (-q)^M}{1+q}.$$

Calculate u_0 for $M = 10$ and $M = 11$ when (i) $q = 0.2$ and (ii) $q = 0.9$. Comment on the difference in the results. Do you have a reasonable explanation for the difference?

We obtain for $M = 10$

$$(i)\ 0.1666 \quad (ii)\ 0.6572$$

and for $M = 11$

$$(i)\ 0.1666 \quad (ii)\ 0.30852$$

No matter whether we have even or odd number of stairs the probability is almost the same (they are not exactly equal) since $q = 0.2$ tells that Per rarely chooses to jump one staircase. On the other hand, if $M = 11$ and q is big, this means that he will probably jump over a staircase and end up treading on $M-1$ with higher probability (i.e. lower for not treading on $M-1$)