Exercise 4.1

Three white and three black balls are distributed equally into two urns. We say that the system is in state i, $i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, we draw one ball from each urn and exchange them. Let X_n denote the state of the system after the n-th step. Explain why X_n is a Markov chain and calculate the transition probability matrix.

Solution: For example, choose $\{X_n, n \geqslant 0\}$ the process such that

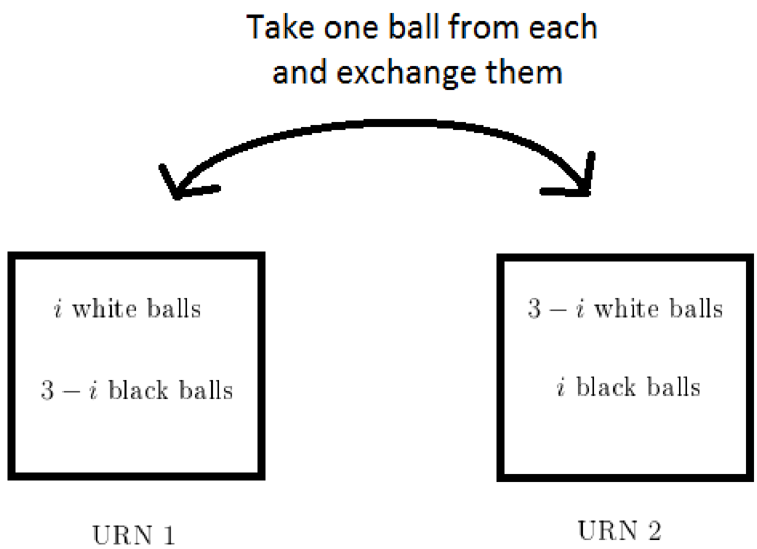
$$X_n = \{\text{Number of white balls in urn 1 after } n \text{ extractions}\}.$$

Clearly, the r.v. X_n determines completely the state of the system at time n. If we know how many white balls there are in urn 1, we know everything. We adopt the standard notation from the book: (X_n satisfies the Markov property, see page 192)

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0) = P(X_{n+1} = j | X_n = i),$$

for all states $i_0, \ldots, i_{n-1}, i, j$ and steps $n \geqslant 0$.

One may easily compute the transition probabilities, for each $n \geqslant 0$. See the following figure to help yourself:



Observe that state X_0 depends on how balls are distributed for the first time. $X_0 \sim Bin(3, 1/3)$. Moreover, observe that

$$P(X_{n+1} = j | X_n = i) = 0 \text{ for all } j < i - 1 \text{ and } j > i + 1.$$

This means that, the probability that we increase/decrease the number of white balls by more than one is 0 in just one step (reasonable).

For $i = 0, 1, 2, 3$ (see the figure above carefully)

$$P(X_{n+1} = j | X_n = i) = \begin{cases} \left(\frac{i}{3}\right)^2 & j = i-1, \\ 2\frac{i}{3}\frac{3-i}{3} & j = i, \\ \left(\frac{3-i}{3}\right)^2 & j = i+1. \end{cases}$$

Observe that $\sum_{j=i-1}^{i+1} P(X_{n+1} = j | X_n = i) = 1$ for all $i = 1, 2, 3$ as it has to be. Using book's notation (recall that $P_{i,j} = 0$ for all i, j s.t. $|i - j| \geqslant 2$)

$$P_{0,0} = 0, P_{0,1} = 1, P_{0,2} = 0, P_{0,3} = 0$$

$$P_{1,0} = 1/9, P_{1,1} = 4/9, P_{1,2} = 4/9, P_{1,3} = 0$$

$$P_{2,0} = 0, P_{2,1} = 4/9, P_{2,2} = 4/9, P_{2,3} = 1/9$$

$$P_{3,0} = 0, P_{3,1} = 0, P_{3,2} = 1, P_{3,3} = 0$$

In matrix form,

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Additional computations: (Very similar to exercise 4.5!) Just for fun, we could be interested in what is the expected number of white balls in the first urn at time $n = 2$. That is, we want to compute $E[X_2]$. We can do so by using:

(1) Knowledge of the starting state X_0 (We know its distribution)

(2) The tower property of the conditional expectation

(3) Chapman-Kolmogorov equation

$$E[X_2] = E[E[X_2|X_0]] = \sum_{i=0}^{3} E[X_2|X_0 = i]P(X_0 = i)$$
$$= \sum_{i=0}^{3}\sum_{j=0}^{3} jP(X_2 = j|X_0 = i)P(X_0 = i).$$

X_0 is the result of distributing 6 balls in two boxes (three white and three black). In other words, it is the result of choosing three balls out of six (white with probability 1/2) and putting them in urn 1. So $X_0 \sim Bin(3, 1/2)$. Hence $P(X_0 = i) = \frac{3!}{(3-i)!i!}(1/2)^3$.

Moreover,

$$P(X_2 = j|X_0 = i) = \sum_{k=0}^{2} P(X_2 = j|X_1 = k)P(X_1 = k|X_0 = i)$$

which using notation from the book (see page 195) is

$$P_{ij}^2 = \sum_{k=0}^{3} P_{ik}P_{kj}$$

also known as, Chapman-Kolmogorov equation.

Hence $P(X_2 = j|X_0 = i)$ the probabilities of transition from i to j in 2 steps are given in the entries of the matrix P^2 which in our case is:

$$P^2 = \begin{pmatrix} 1/9 & 4/9 & 4/9 & 0 \\ 4/81 & 41/81 & 32/81 & 4/81 \\ 4/81 & 32/81 & 41/81 & 4/81 \\ 0 & 4/9 & 4/9 & 1/9 \end{pmatrix}.$$

Using the number from the matrix above and $P(X_0 = 0) = 1/8$, $P(X_0 = 1) = 3/8$, $P(X_0 = 2) = 3/8$ and $P(X_0 = 3) = 1/8$ and probably a bit of patience or a computer we obtain

$$\begin{aligned} E[X_3] = &P(X_2 = 1|X_0 = 0)P(X_0 = 0) + 2P(X_2 = 2|X_0 = 0)P(X_0 = 0) + 3P(X_2 = 3|X_0 = 0)P(X_0 = 0) \\ &+ P(X_2 = 1|X_0 = 1)P(X_0 = 1) + 2P(X_2 = 2|X_0 = 1)P(X_0 = 1) + 3P(X_2 = 3|X_0 = 1)P(X_0 = 1) \\ &+ P(X_2 = 1|X_0 = 2)P(X_0 = 2) + 2P(X_2 = 2|X_0 = 2)P(X_0 = 2) + 3P(X_2 = 3|X_0 = 2)P(X_0 = 2) \\ &+ P(X_2 = 1|X_0 = 3)P(X_0 = 3) + 2P(X_2 = 2|X_0 = 3)P(X_0 = 3) + 3P(X_2 = 3|X_0 = 3)P(X_0 = 3) \\ &= \frac{3}{2}. \end{aligned}$$

This means, if we repeat the experiment many times, in the long-run, there will be "1.5" white balls in URN 1 after two extractions.

Exercise 4.12

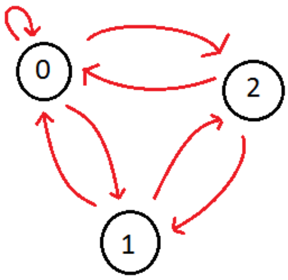
The statement of this exercise is false. We will find a counterexample. One MUST have at least three states. So, $S = \{0, 1, 2\}$ (you can try with two to see why it does not work)

I suggest the following counterexample although this one is not the only one:

Take $X_n : \Omega \to \{0, 1, 2\}$ and **r=2** so the new process with two states is $Y_n : \Omega \to \{0, 1\}$. Then consider as a transition probability matrix P of X:

$$P = \begin{pmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & 0 & p_{12} \\ p_{20} & p_{21} & 0 \end{pmatrix}$$

with none of the p_{ij} either 0 or 1. Then the situation is as follows:



Hence Q (according to the def. in the exercise: $q_{i,j} = \frac{p_{ij}}{1-p_{ir}}$ with $i, j \neq 2$) looks like:

$$Q = \begin{pmatrix} \frac{p_{00}}{1-p_{01}} & \frac{p_{02}}{1-p_{01}} \\ 1 & 0 \end{pmatrix}$$

For example, taking **i=0**, **m=2** and **n=2** we have (look at the diagram to help yourself and think of all the possible paths to follows),

$$P(X_2 = 2 | X_1 \neq 1, X_0 = 0) = \frac{P(X_2 = 2, X_1 \neq 1 | X_0 = 0)}{P(X_1 \neq 1 | X_0 = 0)} = \frac{p_{00}p_{02}}{p_{00} + p_{02}} = \frac{p_{00}p_{02}}{1 - p_{01}}$$

where the last step follows since $p_{00} + p_{01} + p_{02} = 1$.

On the other hand, according to the exercise it should be

$$q_{1,1}^{(2)} = \frac{p_{00}p_{02}}{(1 - p_{01})^2}.$$

Let us check!

$$\frac{p_{00}p_{02}}{1 - p_{01}} = \frac{p_{00}p_{02}}{(1 - p_{01})^2}$$

if, and only if

$$1 - p_{01} = (1 - p_{01})^2$$

if, and only if

$$1 = 1 - p_{01}$$

if, and only if

$$p_{01} = 0!!$$

which can not be since we assumed none of the p_{ij} were 0 or 1. So

$$q_{1,1}^{(2)} \neq \frac{p_{00}p_{02}}{(1 - p_{01})^2}$$

and the formula is wrong.