Exercise 4.20

A transition matrix P is said to be doubly stochastic if the sum over each column equals 1. If such a chain is irreducible and aperiodic and consists of $M+1$ states $0, 1, \ldots, M$ show that the limiting probabilities are given by $\pi_j = \frac{1}{M+1}$ for all $j = 0, \ldots, M$.

Solution: The assumption of irreducibility means that all states communicate with each other and aperiodic means that all states are aperiodic (have period 1), i.e.: For a sufficient large n, $p_{ii}^{(n)} > 0$ for all i. These two assumptions are to ensure that the solution to the system exists and it is unique, see page 216 on top.

The limiting probabilities π_j, $j = 0, 1, \ldots, M$ satisfy

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$$

Now, substituting $\pi_j = \frac{1}{M+1}$ in the above relation, observe that

$$\sum_{i=0}^{M} \frac{1}{M+1} P_{ij} = \frac{1}{M+1} \underbrace{\sum_{i=0}^{M} P_{ij}}_{=1} = \frac{1}{M+1}$$

where we used that $\sum_{i=0}^{M} P_{ij} = 1$ (i.e. the sum over each column is 1).

So $\pi_j = \frac{1}{M+1}$, $j = 0, 1, \ldots, M$ are the limiting probabilities since the solution is unique.

Exercise 4.22

Let Y_n be the sum of n independent rolls of a fair die. Find $\lim_{n\to\infty} P(Y_n \text{ multiple of } 13)$ (in the long run, the proportion of multiples of 13).

Solution: Define a new process $X_n := Y_n \bmod 13$ (this means, we divide Y_n by 13 and take the remainder of the division, of course if the remainder is 0, then Y_n is a multiple of 13). Then X_n has state space $S = \{0, 1, 2, \ldots, 12\}$. X_n is a Markov chain, to know the multiplicity of the next roll it is enough to know the previous one, then the next one is just adding up from 1 to 6. So, for $i = 0, 1, 2, \ldots, 12$ the transition probabilities are

$$P(X_n = (i+k) \bmod 13 | X_{n-1} = i) = \frac{1}{6} \quad \text{for all } k = 1, 2, \ldots, 6$$

and of course $P(X_n = j | X_{n-1} = i) = 0$ otherwise. As we can see, all states are communicative (we can always visit all states) so the matrix is irreducible and all are of period 1. Moreover the matrix is doubly stochastic hence, by exercise 4.20 the limiting probabilities are:

$$\pi = (1/13, \ldots, 1/13).$$

The probability we are looking for is in the first component:

$$\lim_{n\to\infty} P(Y_n \text{ multiple of } 13) = \lim_{n\to\infty} P(X_n = 0) = \pi_0 = \frac{1}{13}.$$

Note: You can surely write down the transition probability matrix of X_n and see why it is dubly stochastic. It looks like:

$$P = \begin{pmatrix}
0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$