In this section we consider the special case where the Markov chain \( \{ X(t) : t \geq 0 \} \), with state space \( \mathcal{X} \), has the property that:

\[
v_i = v, \quad \text{for all } i \in \mathcal{X},
\]

where \( v_i \) as usual denotes the transition rate in state \( i, i \in \mathcal{X} \).

We can then introduce a new process \( \{ N(t) : t \geq 0 \} \), where:

\[
N(t) = \text{The number of transitions in } [0, t], \quad t \geq 0.
\]

It is then easy to see that \( \{ N(t) : t \geq 0 \} \) is a homogeneous Poisson process with rate \( v \).
6.8 Uniformization (cont.)

We then derive an expression for the transition probabilities by conditioning on $N(t)$:

$$P_{ij}(t) = P(X(t) = j|X(0) = i)$$
6.8 Uniformization (cont.)

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$$P_{ij}(t) = P(X(t) = j|X(0) = i)$$

$$= \sum_{n=0}^{\infty} P(X(t) = j|X(0) = i, N(t) = n) \cdot P(N(t) = n|X(0) = i)$$

where $Q_{nj}$ denotes the $n$-step transition probability from state $i$ to state $j$ for the built-in discrete-time Markov chain.
We then derive an expression for the transition probabilities by conditioning on $N(t)$:

$$P_{ij}(t) = P(X(t) = j | X(0) = i)$$

$$= \sum_{n=0}^{\infty} P(X(t) = j | X(0) = i, N(t) = n) \cdot P(N(t) = n | X(0) = i)$$

$$= \sum_{n=0}^{\infty} P(X(t) = j | X(0) = i, N(t) = n) \cdot P(N(t) = n)$$
6.8 Uniformization (cont.)

We then derive an expression for the transition probabilities by conditioning on \( N(t) \):

\[
P_{ij}(t) = P(X(t) = j | X(0) = i)
\]

\[
= \sum_{n=0}^{\infty} P(X(t) = j | X(0) = i, N(t) = n) \cdot P(N(t) = n | X(0) = i)
\]

\[
= \sum_{n=0}^{\infty} P(X(t) = j | X(0) = i, N(t) = n) \cdot P(N(t) = n)
\]

\[
= \sum_{n=0}^{\infty} Q_{ij}^n \cdot \frac{(vt)^n}{n!} e^{-vt}
\]

where \( Q_{ij}^n \) denotes the \( n \)-step transition probability from state \( i \) to state \( j \) for the built-in discrete-time Markov chain.
Since $P(N(t) = n)$ typically is small if $n$ is large, we have the following approximation:

$$P_{ij}(t) \approx \sum_{n=0}^{N} Q_{ij}^n \cdot \frac{(vt)^n}{n!} e^{-vt}$$

provided that $N$ is large.

NOTE: If the built-in discrete-time Markov chain is ergodic, i.e., irreducible, positive recurrent and aperiodic, we have:

$$\lim_{n \to \infty} Q_{ij}^n = \pi_j, j \in X.$$
6.8 Uniformization (cont.)

Since $P(N(t) = n)$ typically is small if $n$ is large, we have the following approximation:

$$P_{ij}(t) \approx \sum_{n=0}^{N} Q_{ij}^n \cdot \frac{(vt)^n}{n!} e^{-vt}$$

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NOTE: If the built-in discrete-time Markov chain is **ergodic**, i.e., irreducible, positive recurrent and aperiodic, we have:

$$\lim_{n \to \infty} Q_{ij}^n = \pi_j, \quad j \in \mathcal{X}.$$ 

Hence, the approximation can be improved by using:

$$P_{ij}(t) \approx \sum_{n=0}^{N} Q_{ij}^n \cdot \frac{(vt)^n}{n!} e^{-vt} + \pi_j \cdot P(N(t) > N).$$
In fact we have:

\[ P_{ij}(t) \approx \sum_{n=0}^{N} Q_{ij}^n \cdot \frac{(vt)^n}{n!} e^{-vt} + \pi_j \cdot P(N(t) > N) \]

\[ = \sum_{n=0}^{N} Q_{ij}^n \cdot \frac{(vt)^n}{n!} e^{-vt} + \pi_j \cdot [1 - P(N(t) \leq N)] \]

\[ = \sum_{n=0}^{N} Q_{ij}^n \cdot \frac{(vt)^n}{n!} e^{-vt} + \pi_j - \pi_j \sum_{n=0}^{N} \frac{(vt)^n}{n!} e^{-vt} \]

\[ = \pi_j + \sum_{n=0}^{N} (Q_{ij}^n - \pi_j) \frac{(vt)^n}{n!} e^{-vt} \]

which typically is a very good approximation even for moderately sized \( N \).
6.8 Uniformization (cont.)

Assume (far) more generally that \( v_i \leq v \) for all \( i \in X \), and let:

\[
Q_{ij}^* = \begin{cases} 
1 - \frac{v_i}{v} & j = i \\
\frac{v_i}{v} Q_{ij} & j \neq i
\end{cases}
\]

The unconditional probability of a transition from state \( i \) to state \( j \) is then \( Q_{ij}^* \).
6.8 Uniformization (cont.)

Assume (far) more generally that $v_i \leq v$ for all $i \in \mathcal{X}$, and let:

$$Q^*_ij = \begin{cases} 
1 - \frac{v_i}{v} & j = i \\
\frac{v_i}{v} Q_{ij} & j \neq i 
\end{cases}$$

$\{X(t) : t \geq 0\}$ can now be interpreted as a Markov chain, where the transition rate is $v$ for all states $i \in \mathcal{X}$. However, only a fraction of the transitions results in actual state changes.
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$\{X(t) : t \geq 0\}$ can now be interpreted as a Markov chain, where the transition rate is $v$ for all states $i \in X$. However, only a fraction of the transitions results in actual state changes.

If the chain is in state $i$, the probability that a transition results in a state change is $v_i/v$, while the probability of no state change is $1 - v_i/v$. 
Assume (far) more generally that $v_i \leq v$ for all $i \in \mathcal{X}$, and let:

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If the chain is in state $i$, the probability that a transition results in a state change is $v_i/v$, while the probability of no state change is $1 - v_i/v$.

Given that a transition results in a state change from state $i$, the probability that the next state is state $j$ is $Q_{ij}$ as before.
Assume (far) more generally that $v_i \leq v$ for all $i \in \mathcal{X}$, and let:

$$Q_{ij}^* = \begin{cases} 1 - \frac{v_i}{v} & j = i \\ \frac{v_i}{v} Q_{ij} & j \neq i \end{cases}$$

{$\{X(t) : t \geq 0\}$} can now be interpreted as a Markov chain, where the transition rate is $v$ for all states $i \in \mathcal{X}$. However, only a fraction of the transitions results in actual state changes.

If the chain is in state $i$, the probability that a transition results in a state change is $v_i/v$, while the probability of no state change is $1 - v_i/v$.

Given that a transition results in a state change from state $i$, the probability that the next state is state $j$ is $Q_{ij}$ as before.

The unconditional probability of a transition from state $i$ to state $j$ is then $Q_{ij}^*$. 
6.8 Uniformization (cont.)

Replacing the $Q_{ij}$s by the $Q_{ij}^*$s in the formula for the transition probabilities, we get:

$$P_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^*^n \cdot \frac{(vt)^n}{n!} e^{-vt}$$

Note that if $v_i = v$ for all $i \in \mathcal{X}$, we get:

$$Q_{ij}^* = \begin{cases} 1 - \frac{v_i}{v}, & j = i \\ \frac{v_i}{v} Q_{ij}, & j \neq i \end{cases} = \begin{cases} 0, & j = i \\ Q_{ij}, & j \neq i \end{cases}$$
Example 6.23

The lifetimes and repair times of a system are independent and exponentially distributed with rates respectively $v_1 = \lambda$ and $v_0 = \mu$. (See Example 6.11.)

The system is modelled as a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with state space $\mathcal{X} = \{0, 1\}$, where$^1$:

$$X(t) = I(\text{The system is functioning at time } t), \quad t \geq 0.$$ 

The matrix of transition probabilities of the built-in discrete-time Markov chain is:

$$Q = \begin{bmatrix} Q_{00} & Q_{01} \\ Q_{10} & Q_{11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$^1$In the Ross(2019) state 0 is the functioning state and state 1 is the failed state.
A uniformized version of this model, is obtained by letting $v = \lambda + \mu$, and:

$$Q_{ij}^* = \begin{cases} 
1 - \frac{v_i}{v} & j = i \\
\frac{v_i}{v} Q_{ij} & j \neq i 
\end{cases}$$
Example 6.23 (cont.)

A uniformized version of this model, is obtained by letting $v = \lambda + \mu$, and:

$$Q^*_ij = \begin{cases} 
1 - \frac{v_i}{v} & j = i \\
\frac{v_i}{v} Q_{ij} & j \neq i
\end{cases}$$

Using that $v_0 = \mu$ and $v_1 = \lambda$, we get:

$$Q^*_00 = 1 - \frac{v_0}{v} = 1 - \frac{\mu}{\lambda + \mu} = \frac{\lambda}{\lambda + \mu}$$

$$Q^*_01 = \frac{v_0}{v} Q_{01} = \frac{\mu}{\lambda + \mu} \cdot 1 = \frac{\mu}{\lambda + \mu}$$

$$Q^*_10 = \frac{v_1}{v} Q_{10} = \frac{\lambda}{\lambda + \mu} \cdot 1 = \frac{\lambda}{\lambda + \mu}$$

$$Q^*_11 = 1 - \frac{v_1}{v} = 1 - \frac{\lambda}{\lambda + \mu} = \frac{\mu}{\lambda + \mu}$$
Example 6.23 (cont.)

In matrix form we get:

\[
\mathbf{Q}^* = \begin{bmatrix}
\mathbf{Q}_{00}^* & \mathbf{Q}_{01}^* \\
\mathbf{Q}_{10}^* & \mathbf{Q}_{11}^*
\end{bmatrix} = \begin{bmatrix}
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu}
\end{bmatrix} = \begin{bmatrix}
a & (1-a) \\
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu}
\end{bmatrix}
\]

where we have introduced \(a = \frac{\lambda}{\lambda+\mu}\).

From this it follows that the 2-step transition probability matrix is:

\[
\mathbf{Q}^{*(2)} = \mathbf{Q}^* \cdot \mathbf{Q}^* = \begin{bmatrix}
a & (1-a) \\
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu}
\end{bmatrix} \cdot \begin{bmatrix}
a & (1-a) \\
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(a + (1-a))a & ((a + (1-a))(1-a)) \\
(a + (1-a))a & ((a + (1-a))(1-a))
\end{bmatrix} = \begin{bmatrix}
a & (1-a) \\
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu}
\end{bmatrix} = \mathbf{Q}^*
\]

Repeating this argument, we get that \(\mathbf{Q}^{*(n)} = \mathbf{Q}^*, \ n = 1, 2 \ldots\)}
Example 6.23 (cont.)

We also recall that:

\[ Q^{*(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \]

Using the formula for the transition probabilities, we get:

\[
P_{ij}(t) = \sum_{n=0}^{\infty} Q^{*n}_{ij} \cdot \frac{(vt)^n}{n!} e^{-vt} = Q^{*0}_{ij} \cdot e^{-vt} + \sum_{n=1}^{\infty} Q^{*n}_{ij} \cdot \frac{(vt)^n}{n!} e^{-vt}
\]

\[
= Q^{*0}_{ij} \cdot e^{-(\lambda+\mu)t} + Q^{*}_{ij} \cdot \sum_{n=1}^{\infty} \frac{((\lambda + \mu)t)^n}{n!} e^{-(\lambda+\mu)t}
\]

\[
= I(i = j) \cdot e^{-(\lambda+\mu)t} + Q^{*}_{ij} \cdot (1 - e^{-(\lambda+\mu)t})
\]
Example 6.23 (cont.)

We then use that:

\[
Q^* = \begin{bmatrix}
Q^*_{00} & Q^*_{01} \\
Q^*_{10} & Q^*_{11}
\end{bmatrix} = \begin{bmatrix}
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\
\frac{\lambda}{\lambda+\mu} & \frac{\lambda+\mu}{\lambda+\mu}
\end{bmatrix}
\]

Inserting this we get:

\[
P_{00}(t) = e^{-\left(\lambda+\mu\right)t} + \frac{\lambda}{\lambda+\mu} \left(1 - e^{-\left(\lambda+\mu\right)t}\right) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-\left(\lambda+\mu\right)t}
\]

\[
P_{01}(t) = \frac{\mu}{\lambda+\mu} \left(1 - e^{-\left(\lambda+\mu\right)t}\right) = \frac{\mu}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-\left(\lambda+\mu\right)t}
\]

\[
P_{10}(t) = \frac{\lambda}{\lambda+\mu} \left(1 - e^{-\left(\lambda+\mu\right)t}\right) = \frac{\lambda}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-\left(\lambda+\mu\right)t}
\]

\[
P_{11}(t) = e^{-\left(\lambda+\mu\right)t} + \frac{\mu}{\lambda+\mu} \left(1 - e^{-\left(\lambda+\mu\right)t}\right) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-\left(\lambda+\mu\right)t}
\]
Example 6.23 (cont.)

We then use that:

\[
Q^* = \begin{bmatrix} Q^*_{00} & Q^*_{01} \\ Q^*_{10} & Q^*_{11} \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\ \frac{\lambda}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}
\]

Inserting this we get:

\[
P_{00}(t) = e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu} \left( 1 - e^{-(\lambda+\mu)t} \right) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}
\]
Example 6.23 (cont.)

We then use that:

\[
Q^* = \begin{bmatrix}
Q_{00}^* & Q_{01}^* \\
Q_{10}^* & Q_{11}^*
\end{bmatrix} = \begin{bmatrix}
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu}
\end{bmatrix}
\]

Inserting this we get:

\[
P_{00}(t) = e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu} (1 - e^{-(\lambda+\mu)t}) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}
\]

\[
P_{01}(t) = \frac{\mu}{\lambda+\mu} (1 - e^{-(\lambda+\mu)t}) = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}
\]
Example 6.23 (cont.)

We then use that:

\[
Q^* = \begin{bmatrix}
Q_{00}^* & Q_{01}^* \\
Q_{10}^* & Q_{11}^*
\end{bmatrix} = \begin{bmatrix}
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu}
\end{bmatrix}
\]

Inserting this we get:

\[
P_{00}(t) = e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu} (1 - e^{-(\lambda+\mu)t}) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}
\]

\[
P_{01}(t) = \frac{\mu}{\lambda+\mu} (1 - e^{-(\lambda+\mu)t}) = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}
\]

\[
P_{10}(t) = \frac{\lambda}{\lambda+\mu} (1 - e^{-(\lambda+\mu)t}) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}
\]
Example 6.23 (cont.)

We then use that:

\[ Q^* = \begin{bmatrix} Q^*_{00} & Q^*_{01} \\ Q^*_{10} & Q^*_{11} \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\lambda + \mu} & \frac{\mu}{\lambda + \mu} \\ \frac{\lambda}{\lambda + \mu} & \frac{\mu}{\lambda + \mu} \end{bmatrix} \]

Inserting this we get:

\[
P_{00}(t) = e^{-(\lambda + \mu)t} + \frac{\lambda}{\lambda + \mu} \left( 1 - e^{-(\lambda + \mu)t} \right) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t}
\]

\[
P_{01}(t) = \frac{\mu}{\lambda + \mu} \left( 1 - e^{-(\lambda + \mu)t} \right) = \frac{\mu}{\lambda + \mu} - \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t}
\]

\[
P_{10}(t) = \frac{\lambda}{\lambda + \mu} \left( 1 - e^{-(\lambda + \mu)t} \right) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}
\]

\[
P_{11}(t) = e^{-(\lambda + \mu)t} + \frac{\mu}{\lambda + \mu} \left( 1 - e^{-(\lambda + \mu)t} \right) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}
\]
Example 6.24

We consider the same two-state system as in Example 6.23, and assume that $X(0) = 1$. We then define:

$$U(t) = \int_0^t X(s)ds = \text{The fraction of the interval [0, t] where } X(s) = 1$$

We can then calculate $E[U(t)]$ as follows:

$$E[U(t)] = E \left[ \int_0^t X(s)ds \right] = \int_0^t E[X(s)]ds$$

$$= \int_0^t P(X(s) = 1|X(0) = 1)ds = \int_0^t P_{11}(s)ds$$
Hence, since we have shown that:

\[ P_{11}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t} \]

we get that:

\[
E[U(t)] = \int_0^t \left[ \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)s} \right] ds \\
= \frac{\mu t}{\lambda + \mu} + \frac{\lambda}{(\lambda + \mu)^2} [1 - e^{-(\lambda + \mu)t}]
\]
Example 6.24 (cont.)

Hence, since we have shown that:

\[ P_{11}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \cdot e^{-(\lambda + \mu)t} \]

we get that:

\[ E[U(t)] = \int_0^t \left[ \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \cdot e^{-(\lambda + \mu)s} \right] ds \]

\[ = \frac{\mu t}{\lambda + \mu} + \frac{\lambda}{(\lambda + \mu)^2} [1 - e^{-(\lambda + \mu)t}] \]

We note that this also implies that:

\[ \lim_{t \to \infty} E \left[ \frac{U(t)}{t} \right] = \frac{\mu}{\lambda + \mu} = \lim_{t \to \infty} P_{11}(t) \]