

Exponential distribution

1. Recap of the previous lecture

$$\lambda > 0.$$

Random variable $X \sim \exp(\lambda) \stackrel{\text{def}}{\iff}$ the density of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} = \lambda e^{-\lambda x} \mathbb{I}_{\{x \geq 0\}}.$$

The cumulative distribution function of $X \sim \exp(\lambda)$

$$F(x) = (1 - e^{-\lambda x}) \cdot \mathbb{I}_{\{x \geq 0\}}.$$

The survival function of $X \sim \exp(\lambda)$:

$$\bar{F}(x) := \mathbb{P}(X > x) = \begin{cases} e^{-\lambda x}, & x \geq 0 \\ 1, & x < 0 \end{cases}.$$

Property

$X_1, X_2, \dots, X_n$ - i.i.d. r.v., $X_1 \sim \exp(\lambda)$

Then

$$Y := \sum_{i=1}^n X_i \sim \text{Gamma}(n, \lambda).$$

2. Minimum of exponential random variables

(2.1) Assume that $X_1 \sim \exp(\lambda_1)$, $X_2 \sim \exp(\lambda_2)$, X_1 and X_2 are independent.

$$\mathbb{P}(X_1 < X_2) = ?$$

Consider a random vector (X_1, X_2) . The density of this r.v. is

$$f_{X_1, X_2}(x_1, x_2) = \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} \mathbb{1}_{\{x_1 \geq 0, x_2 \geq 0\}}$$

$$\begin{aligned} \mathbb{P}(X_1 < X_2) &= \int_0^\infty \int_0^{x_2} \lambda_1 \lambda_2 e^{-\lambda_1 x_1} e^{-\lambda_2 x_2} dx_1 dx_2 = \\ &= \int_0^\infty \lambda_1 \lambda_2 e^{-\lambda_2 x_2} \left(\int_0^{x_2} e^{-\lambda_1 x_1} dx_1 \right) dx_2 = \\ &= \int_0^\infty \lambda_1 \lambda_2 e^{-\lambda_2 x_2} \frac{1}{\lambda_1} (1 - e^{-\lambda_1 x_2}) dx_2 = \\ &= \lambda_2 \int_0^\infty e^{-\lambda_2 x_2} dx_2 - \lambda_2 \int_0^\infty e^{-\lambda_2 x_2} e^{-\lambda_1 x_2} dx_2 = \\ &= 1 - \lambda_2 \int_0^\infty e^{-(\lambda_1 + \lambda_2) x_2} dx_2 \\ &= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2}. \end{aligned}$$

Conclusion

$$\begin{aligned} X_1 &\sim \exp(\lambda_1), \quad X_2 \sim \exp(\lambda_2), \quad X_1, X_2 \text{ are independent} \\ \Rightarrow \quad \mathbb{P}(X_1 < X_2) &= \frac{\lambda_1}{\lambda_1 + \lambda_2}. \end{aligned}$$

(2.2) Assume that $X_1, X_2, \dots, X_n$ are independent and $X_i \sim \exp(\lambda_i)$, $i = 1, \dots, n$.

What is $\mathbb{P}(\min_{i=1, \dots, n} X_i \leq x) \quad \forall x \geq 0$?

Let $x > 0$. Then

$$\begin{aligned} \mathbb{P}\left(\min_{i=1, \dots, n} X_i > x\right) &= \mathbb{P}\left(\bigcap_{i=1}^n \{X_i > x\}\right) = \\ &= \prod_{i=1}^n \mathbb{P}(X_i > x) = \prod_{i=1}^n e^{-\lambda_i x} = e^{-\left(\sum_{i=1}^n \lambda_i\right)x} \end{aligned}$$

Proposition

Let $X_1, \dots, X_n$ be independent random variables such that $X_i \sim \exp(\lambda_i)$, $i=1, \dots, n$. Then

$$\min_{i=1, \dots, n} X_i \sim \exp\left(\sum_{i=1}^n \lambda_i\right).$$

(2.3) Assume that $X_1, \dots, X_n$ are independent r.v. s.t.

$$X_i \sim \exp(\lambda_i), \quad i=1, \dots, n.$$

$$\begin{aligned} \mathbb{P}(X_i < X_j \text{ for all } j \neq i) &= \mathbb{P}\left(\overset{\exp(\lambda_i)}{X_i} < \min_{j \neq i} \overset{\exp(\sum_{j \neq i} \lambda_j)}{X_j}\right) = \\ &= \frac{\lambda_i}{\lambda_i + \sum_{j \neq i} \lambda_j} = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}. \end{aligned}$$

(2.4) The memoryless property of exponential distribution:

$$X \sim \exp(\lambda) \Rightarrow \mathbb{P}(X > s+t \mid X > t) = \mathbb{P}(X > s) \quad \forall s, t \geq 0.$$

Proposition 5.2

Assume that $X_1, \dots, X_n$ are independent and

$$X_i \sim \exp(\lambda_i), \quad i=1, \dots, n. \quad \text{Then } \min_{i=1, \dots, n} X_i \sim \exp\left(\sum_{i=1}^n \lambda_i\right)$$

and, moreover, the rank order of $X_1, \dots, X_n$ is

independent of $\min_{i=1, \dots, n} X_i$.

Proof

The first part regarding $\min_{i=1, \dots, n} X_i$ has already

been proved.

we need to prove

$$\mathbb{P}(X_{i_1} < X_{i_2} < \dots < X_{i_n} \mid \min_{i=1, \dots, n} X_i > t) \stackrel{?}{=} \mathbb{P}(X_{i_1} < X_{i_2} < \dots < X_{i_n})$$

for any permutation $i_1, \dots, i_n$ of $1, \dots, n$.

$$\begin{aligned} \mathbb{P}(X_{i_1} < X_{i_2} < \dots < X_{i_n} \mid \min_{i=1, \dots, n} X_i > t) &= \\ &= \mathbb{P}(X_{i_1} < X_{i_2} < \dots < X_{i_n} \mid \bigcap_{i=1, \dots, n} \{X_i > t\}) = \\ &= \mathbb{P}(X_{i_1} - t < X_{i_2} - t < \dots < X_{i_n} - t \mid \bigcap_{i=1, \dots, n} \{X_i > t\}) \\ &= (\text{property of being memoryless}) = \\ &= \mathbb{P}(X_{i_1} < X_{i_2} < \dots < X_{i_n}). \end{aligned}$$